Middle school students' beliefs about mathematics as a discipline

Ph.D. Dissertation

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Summary

A comprehensive number of research studies point to the centrality of students' beliefs about mathematics in relation to their learning, motivation, and enjoyment of the subject (e.g., Furinghetti & Pehkonen, 2002; McDonough & Sullivan, 2014). Some of these student beliefs are formed early in life and often constitute a filter through which new experiences and information are perceived. Research shows that a *dualistic* view of mathematics, characterized by seeing mathematics as a static body of rules and procedures to be memorized, is not considered conducive for the students' learning or enjoyment of mathematics. In contrast, a *relativistic* view, where mathematics is perceived as a dynamic, coherent and applicable system, is related to high performance, enthusiasm and a positive self-concept (e.g., Gattermann et al., 2012; Grigutsch, 1998).

Developing students' views of mathematics in a relativistic direction thus entails that aspects of mathematics *as a discipline* (understood as the societal, cultural, historical, and scientific aspects of mathematics) are included in mathematics education, thereby contributing to their insights into the nature and role of mathematics. Such insights are central in developing mathematical competence, which is an essential part of their general education (or *Allgemeinbildung*) to be citizens in a democracy. Hence, mathematics education in a democratic society should ideally contribute to such insights. However, there are strong indications of this not necessarily being the case as, for example, pointed out by Schoenfeld (2015) and Fitzmaurice et al. (2021).

The goal of developing beliefs that contribute to the students' mathematical competence entails a discussion about how such beliefs might be developed and investigated. While there is an extensive body of knowledge concerning which beliefs students possess about mathematics, little research has been conducted on the actual *development* of students' mathematics-related beliefs, in terms of how they are formed or changed and how such a development might be initiated, as well as investigated and assessed. Furthermore, only few studies focus on younger students' beliefs, none of them in a Danish context.

The present study focuses on how students' beliefs about mathematics as a discipline may be developed in the context of the Danish compulsory school. In the Danish mathematics curriculum, mathematics as a discipline is represented by the notion of mathematical *overview and judgment*, which in short covers knowledge and beliefs in three aspects: 1. the application of mathematics in other disciplines and fields of practice, 2. the historical development of mathematics, and 3. the nature of mathematics as a subject area (Niss & Højgaard, 2011). There are strong indications that Danish mathematics teachers generally do not consciously consider and apply these aspects in their teaching, perhaps because they do not have the tools for implementing them.

The main purpose of this study is to contribute to the development of the students' beliefs about mathematics as a discipline through an increased focus on the three forms of overview and judgment. It thereby contributes with an understanding of the character of Danish middle school students' beliefs about mathematics as a discipline, as well as of the processes involved in the development of these. Furthermore, it provides knowledge of how beliefs and their development can be investigated and assessed.

The hypothesis of the study is that *a longitudinal change of focus in the teaching of mathematics can contribute to a change in middle school students' beliefs about mathematics—specifically that an increased focus on mathematical overview and judgment can positively influence their beliefs about mathematics as a discipline.* This hypothesis is tested through two research questions:

1. What characterizes Danish middle school students' beliefs about mathematics as a discipline?
2. Which changes can be detected in the students' beliefs about mathematics as a discipline after a longitudinal intervention that focuses on developing the students' mathematical overview and judgment?

Both questions are addressed on two levels: a general level, investigating the beliefs of the students in two 6th grade classes through a questionnaire; and an in-depth level, where the beliefs and attitudes of four of the students are examined through triangulation of data from the questionnaire, interviews and—in relation to research question 2—classroom observations. The first research question focuses on investigating the students' beliefs prior to the intervention. The second research question is addressed through a two-year longitudinal intervention, using a Design-based Research approach to explore which principles of teaching may support the development of the students' overview and judgment. These principles are designed, tested, and adjusted in four iterations in collaboration with the two mathematics teachers of the participating classes. The students' beliefs and their development are analyzed according to their level of reflection, specifically in terms of the level of exemplification, justification, and consistency.

The findings show that although the participating students found mathematics important to learn, their view of mathematics tended to be of a dualistic nature, and their beliefs about mathematics as a discipline were either non-existent or equal to their beliefs about mathematics as a school subject. Furthermore, the level of reflection in their beliefs was generally low. These findings are in line with existing international research. After two years of intervention, the students became more aware of aspects connected to mathematics as a discipline. Their beliefs primarily seemed to have developed in terms of level of reflection—especially within aspects about which the students had not previously developed any beliefs (e.g., the historical development of mathematics). The study thereby shows that peripheral or emerging beliefs are easier to change than beliefs that are based on repeated experiences in the classroom and thus have become central and robust.

In addition, the study's methodological contribution illustrates how the difficulties connected with investigating the complex and inaccessible construct of beliefs may be accommodated through a triangulation of data types, as well as a longitudinal design, facilitated by a Design-based Research approach that enables studying as well as developing the students' beliefs in the context, where they are formed.

The dissertation was carried out in the period February 1, 2019 – August 12, 2022.

Resumé

Adskillige forskningsstudier peger på vigtigheden af elevers forestillinger om matematik i forhold til deres læring, motivation og glæde ved faget (f.eks. Furinghetti & Pehkonen, 2002; McDonough & Sullivan, 2014). En del af disse forestillinger dannes tidligt i livet og udgør dermed et filter, hvorigennem nye erfaringer og informationer fortolkes. Forskning viser, at et *dualistisk* perspektiv, som er karakteriseret ved at opfatte matematik som en statisk samling af regler og procedurer, der skal læres udenad, ikke anses for at fremme elevernes læring eller glæde ved matematik. I modsætning hertil er et *relativistisk* perspektiv, hvor matematikken opfattes som et dynamisk, sammenhængende og anvendeligt system, relateret til høje faglige præstationer, entusiasme og en positiv selvopfattelse (f.eks. Gattermann m.fl., 2012; Grigutsch, 1998).

Aspekter af matematikken *som disciplin* (forstået som matematikkens samfundsmæssige, kulturelle, historiske og videnskabelige aspekter) bør således inddrages i matematikundervisningen for at udvikle elevernes syn på matematik i en relativistisk retning. Dermed øges elevernes indsigt i matematikkens karakter og rolle i verden. Sådanne indsigter er centrale i udviklingen af matematisk kompetence, som er en væsentlig del af deres almendannelse (eller *Allgemeinbildung*) i forhold til at være borgere i et demokrati. Derfor bør matematikundervisningen i et demokratisk samfund ideelt set bidrage til en sådan indsigt. Dog er der stærke indikationer på, at dette ikke nødvendigvis er tilfældet, som for eksempel påpeget af Schoenfeld (2015) og Fitzmaurice m.fl. (2021).

En målsætning om at udvikle forestillinger, der bidrager til elevernes matematiske kompetencer, indebærer en diskussion om, hvordan sådanne forestillinger kan udvikles og undersøges. Mens der findes omfattende viden om, hvilke forestillinger, eleverne besidder om matematik, er der imidlertid

kun en begrænset mængde forskning, der omhandler *udviklingen* af elevers matematikrelaterede forestillinger i forhold til hvordan de dannes eller ændres, og hvordan en sådan udvikling kan igangsættes såvel som undersøges og vurderes. Desuden fokuserer kun få studier på yngre elevers forestillinger om matematik, heraf ingen i en dansk kontekst.

Dette projekt fokuserer på, hvordan elevers forestillinger om matematik som disciplin kan udvikles i den danske grundskole. I Fælles Mål for matematik er matematik som disciplin repræsenteret ved matematisk *overblik og dømmekraft*, der omhandler viden og forestillinger om tre aspekter: 1. matematikkens anvendelse i andre discipliner og praksisfelter, 2. matematikkens historiske udvikling, og 3. matematikkens karakter som fagområde (Niss & Højgaard, 2002). Meget tyder på, at danske matematiklærere generelt ikke medtænker eller anvender disse aspekter i deres undervisning, hvilket muligvis kan relateres til manglende redskaber til at implementere dem.

Hovedformålet med nærværende studie er at bidrage til udviklingen af elevernes forestillinger om matematik som disciplin gennem øget fokus på de tre former for overblik og dømmekraft. Dermed medvirker studiet til en øget forståelse af karakteren af danske mellemtrinselevers forestillinger om matematik som disciplin, samt de processer, der er involveret i udviklingen heraf. Endvidere giver det viden om, hvordan disse forestillinger og deres udvikling kan undersøges og karakteriseres.

Studiets hypotese er, at *et længerevarende fokusskifte i matematikundervisningen kan bidrage til en ændring af mellemtrinselevers forestillinger om matematik – særligt, at et øget fokus på matematisk overblik og dømmekraft kan have en positiv indflydelse på deres forestillinger om matematik som disciplin*. Denne hypotese testes gennem to forskningsspørgsmål:

1. Hvad kendetegner danske mellemtrinselevers forestillinger om matematik som disciplin?
2. Hvilke ændringer kan spores i elevernes forestillinger om matematik som disciplin efter en longitudinal intervention, der fokuserer på at udvikle elevernes matematiske overblik og dømmekraft.

Begge spørgsmål behandles på to niveauer: et generelt niveau, der undersøger elevernes forestillinger i to 6. klasser gennem et spørgeskema, og et dybdegående niveau, hvor fire elevers forestillinger og holdninger undersøges gennem triangulering af data fra spørgeskema, interviews og – i forhold til forskningsspørgsmål 2 – klasserumsobservationer. Det første forskningsspørgsmål fokuserer på at undersøge elevernes forestillinger forud for interventionen. Det andet forskningsspørgsmål addresseres gennem en toårig longitudinal intervention, hvor der anvendes en designbaseret forskningstilgang til at udforske, hvilke principper i undervisningen der kan understøtte udviklingen af elevernes overblik og dømmekraft. Disse principper udformes, testes og justeres i fire iterationer i samarbejde med de to matematiklærere i de deltagende klasser. Elevernes forestillinger og disses udvikling analyseres i forhold til refleksionsniveau, forstået som graden af eksemplificering, argumentation og sammenhæng.

Resultaterne viser, at selvom de deltagende elever fandt det vigtigt at lære matematik, havde deres opfattelse af matematik en tendens til at være af dualistisk karakter, og deres forestillinger om matematik som disciplin var enten ikkeeksisterende eller sammenfaldende med deres forestillinger om matematik som skolefag. Desuden var niveauet af refleksion i deres forestillinger generelt lavt. Disse resultater er i tråd med eksisterende international forskning. Efter to års intervention blev eleverne mere bevidste om aspekter knyttet til matematik som disciplin. Deres forestillinger syntes primært at have udviklet sig med hensyn til refleksionsniveau – navnlig inden for aspekter, som eleverne ikke tidligere havde udviklet nogen forestillinger om (for eksempel matematikkens historiske udvikling). Studiet viser derved, at perifere eller spirende forestillinger er lettere at ændre end forestillinger, der er baseret på gentagne erfaringer i klasseværelset og dermed er blevet centrale og robuste.

Derudover illustrerer projektets metodologiske bidrag, hvordan vanskeligheder forbundet med at undersøge komplekse og utilgængelige forestillinger kan imødekommes gennem en triangulering af datatyper samt et longitudinalt design, faciliteret af en designbaseret forskningstilgang, der muliggør en undersøgelse såvel som en udvikling af elevernes forestillinger i den kontekst, hvor de dannes.

Projektet er gennemført i perioden 1. februar 2019 – 12. august 2022.

Prologue

During the last four years I have, from time to time, been invited to give talks about mathematics anxiety, primarily at teacher conferences. One of my recommendations has been to change the way we look at mistakes, so that students see them as a source for learning instead of a sign of failure.

On a beautiful June evening not long before the submission date of this dissertation, I received an e-mail from an esteemed colleague, whom I have met on one or two of these occasions, as he also gives talks on mathematics education. This e-mail included a link to an interview with the Danish football player Pierre-Emilie Højbjerg, after he and the rest of the Danish national team (with a score of 2–1) defeated the world champions from France with an outstanding performance on the football field in Paris. My colleague wanted to draw my attention to a certain statement from Pierre-Emile, who, according to my colleague, declared that "the best lesson in life is mistakes". My colleague of course sent me this as a curiosity, confirming one of my important points.

I too had, in fact, watched the interview after the match and also noticed this exact statement. Therefore, I was quite certain that the wording was a little different, as I recalled Pierre-Emilie stating that "the best lesson in life is experience". After pointing out this difference, my colleague reheard the interview, and expressed a little disappointment, finding that the clip now had lost its relevance to "our mission". However, in light of this study, Pierre-Emile's actual statement might be even more relevant to the teaching and learning of mathematics. There is certainly wisdom in football.

Copenhagen, July 2022

Maria Kirstine Østergaard

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"The eye sees only what the mind is prepared to comprehend."

— Robertson Davies in the novel "Tempest-Tost", 1951

Chapter 1: Introduction

1.1. Motivation

I began my professional life as a schoolteacher. In social contexts, I often had the same experience when telling people that I taught mathematics: someone (or several) would look repulsed, scared, or appalled, expressing that they either hated math or never had been able to understand it. For some reason, this school subject seemed to evoke negative associations for surprisingly many people—something that I never experienced in relation to any of the other subjects that I taught: English, music, religion, and social studies. My colleagues recounted similar experiences, also only connected to math. I started to wonder what could be the reasons for this. What makes mathematics seem so unpleasant for so many people? Is it particularly hard? Does the subject have an inherent large risk of failure? Is this problem related to the character of the subject or could it be a didactical issue? Is it connected to the way it is taught? When I, later in my professional life, chose a topic for my master's thesis, all these questions led me to investigate the phenomenon of *mathematics anxiety*. By diving into the theories connected to affective factors of mathematics education and by interviewing adults suffering from mathematics anxiety, I discovered the immense importance of beliefs that these adults have developed about mathematics and its teaching and learning (Østergaard, 2018). The four adults that I interviewed generally perceived mathematics in a certain way, where the criteria for success were related to correct results and speed, and where the risk of failure felt huge. While some of them, presumably as a result of their mathematics anxiety, decided that mathematics is irrelevant for daily life, others were convinced that if they only possessed mathematical skills, many of their problems in life would be solved. However, all their negative attitudes towards the subject seemed to relate to experiences in the classroom, and their negative beliefs about mathematics seemed to be connected only to mathematics learned in school.

Existing theory and research are quite clear: students' beliefs about mathematics play an essential part in their learning (e.g., Furinghetti & Pehkonen, 2002; McDonough & Sullivan, 2014). Many students develop a negative relationship to mathematics as early as in elementary school (Boaler, 2016; Ramirez et al., 2013), which often affects their learning and can be difficult to change. Several researchers point to the teaching approach as part of the reason for this (Boaler, 2016; Grigutsch, 1998; Malmivuori, 1996; Schoenfeld, 1992). As exemplified with the four adults that I interviewed, a focus on correct results, speed, and performance seem to be more connected to a perception of mathematics as irrelevant and isolated from the real world than teaching oriented towards mathematical processes (in contrast to products) and application.

However, little research has been conducted on the actual *development* of students' mathematics-related beliefs, in terms of how they are formed or changed and how such a development might be initiated, not to say investigated and measured. On these grounds, I decided to focus my Ph.D. project on developing students' beliefs in a direction that could contribute to their perception of the subject's relevance and its role outside a school context.

More than 40 years ago, Niss (1979) described the *relevance paradox* connected to mathematics education, which classifies the—still existing—fact that while mathematics is objectively perceived as highly relevant, it is often invisible to the individual, and thus subjectively perceived as irrelevant. This invisibility has, partly, to do with the nature of mathematics as often being embedded in or a prerequisite for the matters to which it is applied. As Niss (2021) points out, the role of mathematics in the world is *general* in the sense that it is involved in a substantial diversity of areas and contexts, but without being part of their appearance. If the relevance paradox is to be countered, uncovering the hidden role of mathematics in society and culture is thus central in mathematics education, thereby providing students with nuanced beliefs and knowledge about mathematics.

Mathematics permeates a large variety of activities, fields, and sciences, and for a society, one of the purposes of providing mathematical education for all citizens is thus to ensure that there is adequate mathematical expertise to handle activities, fields, and sciences. Another purpose is to provide the population with general "life skills", enabling them to handle situations and make decisions that involve mathematics. In a democratic society, mathematics education also serves the purpose of preparing its citizens for critical participation and influence (cf. Skovsmose & Nielsen, 1996). Furthermore, current global events have made the centrality of mathematics obvious when describing and handling crises, for example in relation to pandemics, economic crisis, or climate changes (Skovsmose, 2021). Hence, mathematics plays an important role, which cannot be filled by basic mathematical skills. If people are to participate in democratic debate, decision-making, and societal processes, they need insight into the mathematics that is involved in these processes, as well as its role in the structures and mechanisms on which society is built. Moreover, a democracy entails critical questioning of experts and lawmakers, partly to prevent abuse of power. Therefore, citizens in a democracy must obtain a mathematical competence that enables them to exercise these rights and obligations, making mathematics an important part of what in German is called an individual's *Allgemeinbildung*, which—in lack of an appropriate English term—is used to describe a person's life skills, or general education (Biehler, 2019; Winter, 1995).

Insight into the role of mathematics in society, sciences, crisis, and daily life stands in contrast to the subjective perception of mathematics as irrelevant as it is entailed in the relevance paradox. Hence, mathematics education in a democracy should ideally contribute to such insight, but unfortunately, there are strong indications of this not being the case, as for example pointed out by Schoenfeld (2015) and Fitzmaurice et al. (2021).

As will be elaborated later in this dissertation, including aspects of mathematics *as a discipline* in the mathematics education may contribute to developing students' insight into the nature and role of mathematics. "Mathematics as a discipline" is here understood as the scientific nature of mathematics in a broad sense, including both ontological, epistemological, methodological, and utilitarian issues. It thus relates to the societal, cultural, historical, and scientific aspects of mathematics. In the Danish mathematics curriculum for compulsory school, the abovementioned aspects are included in the overall purpose of the subject, and it is emphasized that considerations of these aspects should function as guidelines for the teaching (Danish Ministry of Children and Education, 2019b). In addition, mathematics as a discipline is represented in the—somewhat overlooked—notion of mathematical *overview and judgment*, which in short covers knowledge and beliefs in three areas: the application of mathematics in other disciplines and fields of practice, the historical development of mathematics, and the nature of mathematics as a subject area (Niss & Højgaard, 2011). Mathematical overview and judgment plays a central part in this study and will be elaborated in chapter 2.

1.2. Setting the scene

1.2.1. The Danish school system

In Denmark, public primary and lower secondary education are integrated into a single structure in municipal schools (in Danish: *Folkeskole*). Compulsory education begins at age six with one year of pre-school (grade 0), and nine school years (grades 1-9), with a possibility of prolonging the compulsory education for one, optional year (grade 10).

Following compulsory schooling, the students can choose their educational path. General upper secondary education is divided into four programs: Higher General Examination (STX), Higher Technical Examination (HTX), Higher Commercial Examination (HHX), and Higher Preparatory Examination (HF). In addition, students can choose a vocational upper secondary education. Where vocational education qualifies for the labor market, general upper secondary education primarily prepares students for higher education, which can take place at different educational institutions, e.g., business academies, professionally oriented university colleges or universities. Teachers in grades 1-9 of compulsory school are educated at university colleges, in a four-year program oriented directly towards teaching in primary and lower secondary school, which provide them with a bachelor's degree in education. All teachers specialize in approximately three subjects, which means that mathematics teachers have an education that integrates mathematics, mathematics education and teacher practice. On upper secondary level, teachers are educated at the university in a five-year master's program.

The present study concerns students in middle school. It takes place in two 6th grade classes, which means that the students were approximately 12 years old at the beginning of the study. The classes thus follow the curriculum for compulsory school, which covers grades 1-9. It determines regulations and so-called "Common Objectives" (in Danish: *Fælles Mål*) within each subject for specific grade levels (grades 3 and 6), and end levels (grade 9) (Danish Ministry of Children and Education, 2019a). These "Common Objectives" define the direction and goals of the teaching. Competence goals are binding national objectives, with a guiding set of knowledge and skills aims, whereas the individual school can formulate local curricula and descriptions of educational ways to reach the grade level and end level objectives or choose to follow the national guidelines. Local curricula must be approved by the municipal board, thus becoming binding. As the objectives and the curriculum are quite general, the teachers have a high degree of autonomy and can choose their own materials, textbooks, teaching approach, methods etc.

For the subject of mathematics, the "Common Objectives" are largely built upon two strands: mathematical competencies and mathematical content areas. Overall, the students are expected to develop mathematical, competencies, skills, and knowledge and to apply these in real situations. In section 2.3.1, the curriculum for mathematics is elaborated.

1.2.2. The research field of beliefs

At the beginning of the twentieth century, the concept of beliefs mainly belonged in the field of social psychology (Furinghetti & Pehkonen, 2002; Thompson, 1992), where it served to study what might influence people's actions. The concept was then, to some extent, forgotten with the increased focus on more observable and measurable aspects of human behavior that followed with behaviorism, but the interest reemerged in the 1960s and 1970s in connection with cognitive science (Abelson, 1979). The interest in studying beliefs grew larger in the 1980s in various disciplines, e.g., psychology, political science, and anthropology. In the context of educational science, a shift in paradigms from teachers' *behavior* to teachers' *thinking and decision-making processes* contributed to the renewed interest in the concept of beliefs (Thompson, 1992). This was particularly seen in the late 1980s when challenges, in relation to the implementation of a more problem-oriented teaching approach, in the United States were thought to be connected to the teachers' "inappropriate" beliefs about the subject of mathematics and about mathematical problem solving (Rösken et al., 2011; Schoenfeld, 2007b).

A large part of the research on beliefs within the field of mathematics education has thus focused on teachers' beliefs (and still does), not only about mathematics as a subject but also about mathematics teaching and learning (e.g., Philipp, 2007; Richardson, 1996; Thompson, 1992, just to mention a few essential contributions). Another, but later, focus has been on *students'* beliefs, as it became widely acknowledged that affective factors (Hannula, 2011) have an essential influence on students' learning (e.g., Grootenboer & Marshman, 2016; McDonough & Sullivan, 2014; Op't Eynde et al., 2002; Schoenfeld, 1985). In 2002, the influential book "Beliefs—a hidden variable in mathematics education?" (Leder et al., 2002) was published based on the presentations and discussions at an international meeting concerning mathematics-related beliefs in Oberwolfach in 1999. To some extent, the background for this meeting was a concern about students' decreasing interest in mathematical education, which initiated a focus on students' thoughts, perceptions, and feelings about the subject. A section on students' beliefs was thus included in the book, along with sections concerning teachers' beliefs (as well as conceptualization and measurement of beliefs).

Research studies have suggested that students' beliefs influence their behavior, for example in relation to problem solving (Schoenfeld, 1992), their emotional responses (Op't Eynde et al., 2002), their decisions, as well as their interest, and motivation (Kloosterman, 2002), which then, in turn, affect their level of engagement in a classroom. Several researchers have also emphasized that while beliefs influence learning, experiences in the classroom conversely influence students' beliefs (e.g., Garofalo, 1989; Grootenboer & Marshman, 2016; Schoenfeld, 1988). Thereby, students' beliefs become part of the psychological context of their mathematical learning (Schoenfeld, 1985).

Hence, the study of students' beliefs in mathematics education is—still—highly relevant. As Green states: "It seems intuitively obvious, […] that the acquisition of beliefs or their modification is a major concern in the activity of teaching" (Green, 1971, p. 42). In the words of Schoenfeld (2015):

> *[…] understanding why people (specifically, students and teachers) do what they do is what really counts. If we want students to become effective problem solvers, we need to know why they make the productive or unproductive choices they make while engaged in problem solving. If we want teachers to become more effective at producing students who are good mathematical thinkers, we need to know why they make the productive or unproductive choices they make while engaged in teaching. Needless to say, the study of beliefs is central to this issue (p. 397).*

Or as Törner (2014, p. 11) points out: "It is very complicated to avoid beliefs. We have to be aware of that!"

However, the concept of beliefs is ambiguous. Its definition as well as its relation to other cognitive and affective aspects of mathematics learning and teaching are the subjects of currently ongoing discussions (e.g., Furinghetti & Pehkonen, 2002; Leder, 2022; Pajares, 1992), and various frameworks exist and emerge (e.g., Hannula, 2012; Op't Eynde et al., 2002). Empirical research on students' mathematics-related beliefs is for a large part addressed through descriptive studies, seeking to identify their character and content (e.g., Grootenboer & Marshman, 2016; Rolka & Halverscheid, 2011). Investigating issues related to the development or change of students' beliefs is not as common, also since it is generally agreed that these processes are difficult and time consuming (Green, 1971). Hence, there exists an extensive body of knowledge concerning which beliefs students possess about mathematics, but only limited knowledge concerning how they can be changed or modified.

1.3. Relevance in a Danish and an international context

The mathematics-related beliefs of Danish middle school students have, to my knowledge, not yet been investigated from a research point of view. There are, however, Danish research studies pointing to the nature of factors that might influence their beliefs, for instance, the kind of typical activities in the classroom. For example, Bremholm et al. (2016), find that the majority of mathematical tasks presented in 4th grade classes at 14 Danish schools do not require reflected decision making or creative production, and that the teaching to a large extent is organized as individual work (Bundsgaard & Hansen, 2018, same study). Thereby, the teaching may influence the students' beliefs in a way that increases a result- and formula-oriented perception of mathematics, which both Grigutsch (1998) and Malmivuori (1996) show is related to a lack of enthusiasm, poor performance and poor assessment of one's own performance. This indicates that this part of the teaching in school mathematics seems to contribute to a development of students' mathematics-related beliefs in a direction that does not support the students' learning in terms of a deep understanding of mathematical relations and structures. Since such inappropriate beliefs seem to be self-reinforcing (Østergaard, 2018), it is relevant to investigate how such a development can be avoided or reversed early in the education system. This is supported by research literature showing that students' enjoyment of mathematics decreases with age (Hannula et al., 2018), and that a negative attitude towards the mathematics subject may become more prevalent with age, even in primary school, as found in a study by Blomqvist et al. (2012) among students in 2nd and 5th grade. Conversely, Wæge (2008) as well as Grigutsch (1998) find that a process- and application-oriented view of mathematics is associated with enthusiasm, good performance and a positive professional self-image.

An orientation towards mathematical processes and applications may thus be relevant in the discussion concerning the development of beliefs about mathematics as a discipline, which (as mentioned) is highly present in the overall purpose of the subject of mathematics in the Danish curriculum, and specifically represented in the notion of mathematics overview and judgment. Although the national guidelines for the Danish mathematics curricula suggest that the students' overview and judgment is developed throughout compulsory school (Danish Ministry of Children and Education, 2019b, p. 10), the binding competency goals in the "Common Objectives" only mentions this notion as part of overall aims. It is thus up to the teachers to decide how these goals and aims are obtained in practice. Judging from the findings of the abovementioned research, as well as from my own experience with mathematics education, there are strong indications that these goals are generally not explicit in the average teachers' approach to mathematics and that (although the teachers are aware of these goals) they might not consciously consider them. Furthermore, it seems that perhaps the teachers in general do not have the tools for implementing these goals.

In 2015, Jankvist (2015a, 2015b) successfully investigated how a focus on overview and judgment may potentially lead to changes in upper secondary school students' beliefs about mathematics as a discipline. The present study thus builds on these results, as its overall focus resembles the one of Jankvist's study. However, this study takes into account that students' beliefs about mathematics—as mentioned above—are established in the early school years based on experiences in the classroom. Therefore, it might be argued that it is appropriate to address the development of students' overview and judgment earlier than upper secondary school. By specifically focusing on the students' overview and judgment in middle school, the ambition of this study is partly to begin this development sooner and thus highlight the role of mathematics in the world, and partly to contribute to a positive attitude towards the subject, as well as to increase the students' perception of its relevance.

As pointed out by Jankvist (2015a), beliefs are often perceived as a means, for instance, to understand and explain students' approaches to mathematics or to improve their learning (e.g., Grigutsch, 1998; Rösken et al., 2011; Schoenfeld, 1992), but the development of students' beliefs about mathematics may also be seen as a goal (in the sense of a general purpose of education) in themselves (as for example seen in Greer et al., 2002; Yackel & Rasmussen, 2002). In light of the above-mentioned purposes of mathematics education, there might be good reasons for considering beliefs from this more normative perspective. An example of this is partly found within a publication on the 2012 Programme for International Student Assessment (PISA), where the importance of affective factors is recognized:

> *Individuals' attitudes, beliefs and emotions play a significant role in their interest and response to mathematics in general, and their employment of mathematics in their individual lives. Students who feel more confident with mathematics, for example, are more likely than others to use mathematics in the various contexts that they encounter. Students who have positive emotions towards mathematics are in a position to learn mathematics better than students who feel anxiety towards that subject. Therefore, one goal of mathematics education is for students to develop attitudes, beliefs and emotions that make them more likely to successfully use the mathematics they know, and to learn more mathematics, for personal and social benefit. (OECD, 2013, p. 42)*

The goal of developing beliefs that contribute to the students' mathematical competence as part of their *Allgemeinbildung* and democratic citizenship entails a discussion about how such beliefs might be developed and investigated (Jankvist, 2015a). Research on development in general, and on beliefs in particular, can hardly be conducted through descriptive studies, which, as mentioned, constitute the large majority in the field. In order to understand the development of beliefs, researchers call for longitudinal studies (e.g., Hannula et al., 2018), taking place within the context of where the beliefs are developed (Goldin et al., 2016). The present study accommodates this need by approaching the students' beliefs through a two-year, longitudinal intervention conducted in the classroom. It may thereby contribute to the field of mathematics education in terms of:

- An understanding of the character of Danish middle school students' beliefs about mathematics as a discipline.
- An understanding of the processes involved in the development of students' beliefs about mathematics (as a discipline).
- An increased knowledge about accessing students' beliefs.
- An increased knowledge about investigating, not only students' beliefs, but also their development.
- An understanding of methodological approaches and concerns connected to developing, accessing, and assessing students' beliefs.

1.4. Hypothesis

Based on these preliminary considerations concerning the relevance of studying students' beliefs about mathematics as a discipline in a Danish context, my hypothesis is that:

A longitudinal change of focus in the teaching of mathematics can contribute to a change in middle school students' beliefs about mathematics – specifically that an increased focus on mathematical overview and judgment can positively influence their beliefs about mathematics as a discipline.

In particular, an articulation of the application, historical development, and nature of mathematics is expected to contribute to the students' beliefs, including the relevance of the subject.

1.5. Research questions

1.5.1. Research question 1

To study the development of students' beliefs, it is necessary to investigate the character of their beliefs before the intervention. This will not only provide information on their pre-beliefs (and thereby a ground for comparison), but also contribute to the design of the intervention as well as to an understanding of how middle school students generally perceive the subject. The first research question is thus:

1. **What characterizes Danish middle school students' beliefs about mathematics as a discipline?**

1.5.2. Research question 2

The second research question focuses on the development of the students' beliefs about mathematics as a discipline through an intervention that focuses explicitly on developing the students' overview and judgment—an already existing, but overlooked, part of the Danish mathematics curriculum.

2. **Which changes can be detected in the students' beliefs about mathematics as a discipline after a longitudinal intervention that focuses on developing the students' mathematical overview and judgment?**

1.6. Research strategy

Research question 1 is researchable in the sense that it can be investigated through a structured process of data collection and analysis among the 43 (38) participating middle school students in two 6th grade classes (5 students left the school during the project). As beliefs are part of complex cognitive and affective structures and can only be approached through interpretive methods, they are not easily investigated. Furthermore, they are potentially influenced by the context in which they are evoked. To accommodate this complexity, I address the research question on two levels:

1. An overall level, where general tendencies in the students' beliefs were assessed through a pre-questionnaire among *all* the students in the two participating classes. The questionnaire consisted of mixed types of items, designed to investigate primarily their beliefs about the three forms of overview and judgment, and secondarily their attitudes towards mathematics.
2. An in-depth level, where four selected students were interviewed and asked to elaborate on their beliefs about mathematics as a discipline, in addition to other aspects that may influence their beliefs: attitude towards mathematics, learning behavior and beliefs about mathematics education, themselves as learners of mathematics and the classroom context.

Research question 2 builds on the findings from research question 1, as it provides information about the starting point of the development of the students' beliefs. To answer this question, it is necessary to also address methodological issues concerning how changes in beliefs can be investigated and assessed[1], and in which ways classroom teaching can support or contribute to the development of overview and judgment. Through a two-year longitudinal intervention, I used a Design-based Research (DBR) approach to investigate which principles of teaching (i.e., means and mechanisms) that can support students' development of the three forms of overview and judgment. In collaboration with the teachers of the two participating classes, I developed these teaching principles by designing, testing, and adjusting them through four DBR-iterations. Like the first research question, the second is researchable, although it is a complex matter, in the sense that it is possible to collect and analyze data on students' beliefs and examine their development through comparative analyses. The specificity and complexity of the issues raised are considered appropriate to investigate through a study of this format and within the possibility and limits of a Ph.D. project. The research question thus appears feasible within the timeframe and practical constraints. It is addressed on the same two levels as the former:

1. Overall level: All participating students responded to a post-questionnaire after the completion of the intervention. The post-questionnaire was, with a few exceptions, similar to the pre-questionnaire, which made it possible to compare the answers given before and after the intervention, and thus investigate the development of the overall tendencies in the students' beliefs.
2. In-depth level: The four selected students were re-interviewed on two occasions: halfway through the intervention and at the end of the intervention. In addition, the teaching was observed and video recorded approximately two or three times a month during the two years of intervention. During these observations, special attention was paid to the actions, behavior, and statements of the four students. Furthermore, the recordings served the purpose of documenting the teaching and learning activities. The three types of data concerning the focus students' belief-development (questionnaires, interviews and video-observations) were triangulated, analyzed and compared to the findings of research question 1 (i.e., the students' pre-beliefs).

1.7. Overview of the dissertation

Chapter 2 presents the theoretical concept of beliefs, both as a general concept in research and in relation to research in mathematics education. I begin section 2.1 by giving a conceptual clarification, not only of beliefs, but also of the affective domain and concepts related to beliefs. I then discuss the definition of beliefs and address how they are formed and structured, and how they can be investigated

[1] The term 'assess' might be associated with formal assessment in the sense of grade, judge, or value. However, I here, use the term in the sense of investigate, examine, document, or review.

and measured. Next, I turn to students' beliefs about mathematics as a discipline in section 2.2, partly addressing this matter through a literature review. This review not only helps illustrating the relevance of the overall project, but also informs the design of the intervention, and contributes to a framework for analyzing the students' beliefs.

One of the key aspects of the project is the notion of mathematical overview and judgment, which is elaborated and put into the context of the Danish mathematics curriculum in section 2.3. Section 2.4 addresses how beliefs are developed or changed, including a discussion of the role of reflection in this process. Finally, section 2.5 discusses if the "ideal" beliefs can be defined and, if so, what characterizes them. This discussion is both related to students' beliefs about mathematics as a discipline in general and in relation to the aims of this particular project and intervention.

Chapter 3 describes the methodology of the study. The first section (3.1) elaborates the philosophical worldview, which is characterized by a constructivist approach to research. Next, I explain the study's strategy for inquiry in section 3.2, focusing primarily on the features of Design-based Research and their connection to the research questions and aims of this study. The criteria on which the quality of the study can be valued are elaborated in section 3.3, including considerations on ethical issues.

Subsequently, I classify the empirical methods used to address the research questions in **chapter 4**. The content of this chapter includes descriptions and considerations concerning data collection and triangulation of different data types (questionnaire, interview, and classroom observations), as well as strategies for data analysis. The choices behind these methods are elaborated and related to the theoretical and methodological perspectives of the study.

Having established the theoretical and the methodological foundation, I turn to the empirical part of the study in **chapter 5**. It is connected to research question 1, investigating what kind of pre-beliefs the participating students have about mathematics as a discipline. Thus, data from the beginning of the intervention is presented and analyzed. The chapter includes a brief and general illustration of the beliefs of all the students and a more thorough and detailed analysis of four selected students' beliefs. A subdiscussion of the findings of research question 1 is found in **chapter 6**.

These findings play an important role in the design of the intervention, through which research question 2 is addressed. **Chapter 7** describes this design and elaborates the implemented teaching principles, the four iterations of the study, and the adjustments made along the way. In addition, I present examples of teaching activities used during the intervention.

According to chapter 5, which addresses research question 1, the second research question is addressed in **chapter 8**, where data from the intervention are presented, analyzed, and compared to the data connected to research question 1. The chapter considers possible changes in the students' beliefs, both for the entire group of students and in particular for the four focus students, who are presented as cases. **Chapter 9** is a subdiscussion of the findings related to research question 2.

The overall study and its key findings are discussed in **chapter 10**, which includes considerations of the implications and the contributions in a theoretical and a methodological perspective. Furthermore, section 10.3 addresses issues connected to the role of the teachers in the study by drawing on theory on implementation of research, as well as on teachers' beliefs. In section 10.5, the study is evaluated based on the quality criteria presented in chapter 3, and implications for practice are considered in section 10.6.

Finally, I summarize and present the conclusions of the study in **chapter 11**.

1.8. Who might benefit from reading this dissertation?

This dissertation may be of relevance to several groups of readers. One of them is, of course, fellow researchers of mathematics education, especially within the field of affective aspects of mathematics learning. The study provides new perspectives, considerations, and discussions to the field, thus potentially contributing to future research and nuancing existing research.

As the study concerns students' beliefs, the dissertation may also benefit both pre- and in-service mathematics teachers, particularly in relation to the importance and influence of students' beliefs about mathematics as a discipline and the process of developing such. In addition, the dissertation includes thoroughly described suggestions for an approach to competence-oriented teaching, with a special focus on the students' mathematical overview and judgment. Students' beliefs about mathematics are largely formed and developed in the classroom, and, needless to say, the teachers play a central role in this process.

Another influential factor related to the development of students' beliefs in the classroom is the teaching materials. Textbooks, digital learning platforms, etc. contribute to certain representations and images of mathematics, which may influence the students' experiences with and perception of mathematics. Hence, editors, developers, and authors of these materials may also find this study and its findings useful and relevant for their general approach to communication of mathematics.

Finally, the dissertation may be of interest to policymakers, especially in regard to potentially implementing an increased focus on mathematical overview and judgment and acknowledging the essentiality of students' beliefs in relation to their learning, their motivation, their perception of relevance, their mathematical competence, their *Allgemeinbildung*, and their possibilities and obligations in a democratic society.

Although it is a rather ambitious aim, it might be argued that anyone who, in one way or another, potentially influences students' beliefs about mathematics (and has the stamina and urge to read a full Ph.D. dissertation), would benefit from reading along. This group includes pedagogues, parents, authors of children's literature, makers of TV-shows, movie producers, etc. Children's and students' beliefs about mathematics are formed through their experiences and encounters with mathematics even before entering school, acting as filters for new experiences, and thus shaping their learning, enjoyment, and approach to mathematics throughout their lives. An increased awareness on all levels (maybe in particular on a pre-school level and in the early school years) of the importance of the way mathematics is represented and which beliefs we all contribute to the development of may, in fact, make a difference to future students.

Chapter 2: Mathematics-related beliefs

This chapter focuses on the theoretical aspects of beliefs. In the following section, I first conceptually clarify the concept of beliefs and related concepts. Here, the concept is not related to students or teachers in particular, but treated as an overarching and general concept, drawing on research on both areas. I then present considerations connected to the definition, formation, structure, and measurement of beliefs. In section 2.2, I focus specifically on students' beliefs about mathematics as a discipline, partly through a review of existing literature on this issue. The notion of *mathematical overview and judgment* is presented in section 2.3. This notion is used as a representation of the belief dimension related to 'mathematics as a discipline', not only in the intervention of the study but also in the analyses of the data. In section 2.4, I present theoretical considerations as well as existing empirical results connected to the endeavor of changing, developing, or modifying (students') beliefs. Special attention is given to the central role of *reflection*. Section 2.5 contains considerations related to whether the 'ideal' beliefs can be identified or defined.

Although my focus is on students' beliefs, I also draw on theory related to teachers' beliefs, based on the following two considerations. First, the research literature on teachers' beliefs is more comprehensive, as this area has been a research subject longer than students' beliefs have. Furthermore, there are certain challenges connected to investigating students' beliefs, which makes research more difficult. For example, teachers are more likely to be able to express their beliefs than (young) students, and a teacher's actions in a classroom is perhaps more easily interpreted as expressions of beliefs. Second, in many aspects, beliefs are beliefs, independent of the beholder's age and occupation. Theoretical considerations concerning, for example, conceptual definition, structural aspects, or the processes involved in the formation and change of beliefs can be said to be equivalent regardless of the subject holding the beliefs. With that said, there are, of course, aspects and issues where it is appropriate to distinguish between students' and teachers' beliefs, for example when looking at the dimensions included in a person's mathematics-related belief system. Here, a teacher's beliefs inevitably include beliefs that are not part of a student's beliefs (e.g., about mathematics teaching and learning). Investigating teachers' and students' beliefs may require different approaches, and the expressions of beliefs as well as what is considered "appropriate" may also vary. Hence, sections concerning the general concept of beliefs include both areas, while I in other sections specifically distinguish between students' and teachers' beliefs when it is relevant.

2.1. The concept of beliefs

Generally, the literature points to the ambiguous use of the concept of beliefs (e.g., Furinghetti & Pehkonen, 2002; Goldin et al., 2009; Leder, 2022; Rolka & Roesken-Winter, 2015). This diversity in the understanding and application of the concept can be quite consequential in relation to the research in the field and how research results are received (Di Martino & Zan, 2011). As stated by Pajares (1992) and Thompson (1992), as well as by most recent belief-related academic literature, there is no consensus on a definition of the concept of beliefs, and no uniquely determined theory (Törner, 2014). A small but frequently cited study by Furinghetti and Pehkonen (2002) shows the diversity in the perception of the concept among several mathematics education specialists.

There are numerous associated concepts and synonyms related to beliefs. Pajares (1992) presents a long list of these, and Mason (2004) produces "an entire alphabet of associated interlinked terms":

> *A is for attitudes, affect, aptitude, and aims; B is for beliefs; C is for constructs, conceptions, and concerns; D is for demeanor and dispositions; E is for emotions, empathies, and expectations; F is for feelings; G is for goals and gatherings; H is for habits and habitus; I is for intentions, interests, and intuitions; J is for justifications and judgements; K is for knowing; L is for leanings; M is for meaning-to; N is for norms; O is for orientations and objectives; P is for propensities, perspectives, and predispositions; Q is for quirks and quiddity; R is for recognitions and resonances; S is for sympathies and sensations; T is for tendencies and truths; U is for understandings and undertakings; V is for values and views; W is for wishes, warrants, worlds, and weltanschauung; X is for xenophilia (perhaps); Y is for yearnings and yens; and Z is for zeitgeist and zeal. (p. 347)*

Furinghetti and Pehkonen (2002) mention that the placement of beliefs in relation to cognitive and affective issues is one of the central objects for diverse comprehensions. Some researchers perceive beliefs as part of the affective domain, as for example McLeod (1989)who, in his subsequent framework (1992), divides affect into three dimensions: beliefs, attitudes and emotions. Others see beliefs as primarily cognitive, as for example Yackel and Rasmussen (2002), who view beliefs as being the psychological equivalent to norms. Also Furinghetti and Pehkonen (2002) and Grigutsch (1998) place beliefs as part of the cognitive domain, namely as subjective knowledge, but with a close connection to the affective domain. Several researchers view beliefs as a bridge between these two domains (e.g., Grootenboer & Marshman, 2016; Schoenfeld, 1985).

Considering this ambiguity, Furinghetti and Pehkonen (2002) recommend that:

> *[…] authors of studies on beliefs reduce the terms and the concepts involved in their work to the minimum needed. Additionally, they have to make clear their assumptions, the meaning they give to basic words, and the relationship between the concepts involved. (p. 55)*

Following this recommendation, I investigate the diversity in conceptualization in the following section, where I elaborate on a few of the associated concepts, their interrelation, and their relevance to this project.

2.1.1. Conceptual clarification: the affective domain, attitudes, and knowledge

As an introduction to this section, I give a brief overview of the affective domain to locate the approach of this project, paying special attention to the concept of attitudes and the relationship between beliefs and the cognitive concept of knowledge.

The affective domain in mathematics education research

The importance of affect in students' learning of mathematics is indisputable. In a learning situation, factors such as motivation, emotions, attitudes, social norms, and beliefs may all influence a students' engagement, participation, perseverance, performance, etc. Within mathematics education research, the affective domain is typically perceived as the non-cognitive aspects of human thought (Hannula, 2012), although cognitive aspects are included in research on, for example, beliefs or motivation. The affective domain is to a large extent used as a general or overall term (Hart, 1989), but its components and their relations have been described and characterized in several ways, of which I shall elaborate on a few (*attitudes* and *emotions*) while paying special attention to the employment of 'beliefs'.

A meta-theoretical framework for the affective domain (figure 1) was constructed by Hannula (2011, 2012), partly based on a discussion taking place at CERME5 (Hannula et al., 2007) regarding the complexity of this domain. Here, it was argued that the affective domain includes not only factors related to emotion, motivation, and cognition, but also the contexts in which affect is formed and developed, such as the classroom or the socio-historical contexts. Hannula's framework takes this into account by describing three distinct dimensions for identification and definition of theoretical concepts within the affective domain:

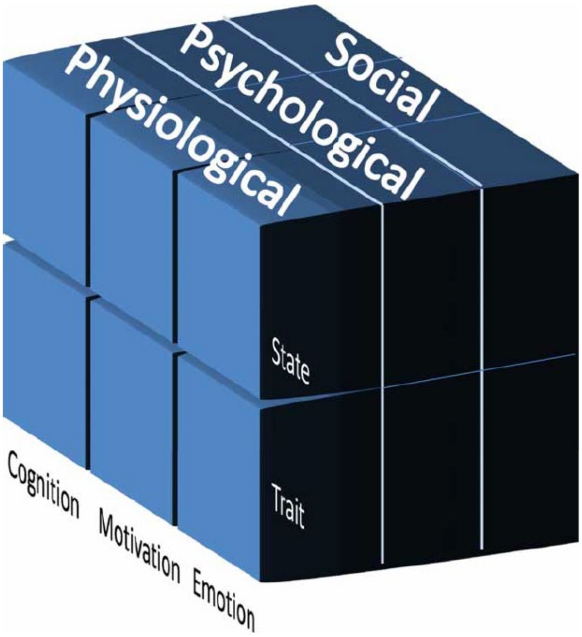


Figure 1: Dimensions for affective constructs (Hannula, 2012, p. 144)

- Stability: Concepts can be of a stable nature (*trait*), or a dynamic nature (*state*). For example, emotions are often characterized as relatively unstable; yet some emotions may be said to be stable in similar situations (e.g., *frustration* when presented with equations). Likewise, beliefs are considered relatively stable, but Hannula (2011) argues that beliefs also have a state aspect, for example when a student with low self-efficacy realizes that (s)he is in fact able to solve a mathematical task.
- Elements: Concepts can be *cognitive* (what one believes), *emotional* (what one feels), or *motivational* (what one desires). The cognitive element includes "mental representations to which it makes sense to attribute a truth value" (Hannula, 2011, p. 43). The purpose of this dimension is not only to distinguish these three aspects, but also to focus on the relationships between them. For example, beliefs may be reflected in a person's emotions, or a motivational goal related to performance may influence one's beliefs about mathematics learning.
- Processes: Concepts can be of a *physiological* (e.g., neurological functions or—related to cognitive elements—bodily expressions involved in thinking processes, such as talking and gesturing), *psychological* (e.g., attitudes, emotions, and beliefs) or *social* nature (e.g., norms, social interactions, or classroom climate).

The framework and its three dimensions are primarily meant as a tool for categorizing and relating theories and research, and not for empirical analysis. For this study, I apply the often-cited framework for the affective domain by McLeod (1992), which is based on a review of research on affect in mathematics education. McLeod identifies three dimensions in the affective domain: *Beliefs, attitudes,* *and emotions*. In this order, the dimensions decrease in both stability and cognitive structure and increase in intensity. Thus, beliefs are more stable and resistant to change than attitudes and emotions, and they develop slowly over a long period of time, but they are felt less intensely and are more cognitive in nature than attitudes and emotions. Below, I characterize McLeod's framework according to Hannula's dimensions and argue for the choice of this framework in relation to this study.

- Stability: In McLeod's framework, beliefs are primarily seen as a concept of trait aspects, and emotions as a concept of state aspects. Considering Hannula's argument about stable emotions, one might also argue that reoccurring emotions in similar situations may be expressions of stable beliefs, which are connected to those specific situations. Likewise, it might be argued that a lasting change in beliefs, for example about one's self-efficacy, probably requires repeated experiences of success, and thereby the level of state aspects of beliefs is low. As the focus of this study is to develop students' beliefs, it might be purposeful to perceive them as primarily a trait aspect, as this entails a longitudinal and consistent intervention with repeated experiences. A state aspect might result in an inadequate intervention based on only a few experiences.
- Elements: McLeod's three dimensions are placed within a spectrum of cognition and emotion. Thereby, McLeod does not explicitly include the element of motivation in his framework (which is one of Hannula's arguments for constructing a new framework). McLeod sees emotions as initiated by beliefs, and attitudes originating from repeated emotional reactions. In relation to this study, this framework emphasizes the interaction between cognition and affect, which is important in learning processes.
- Processes: Beliefs, attitudes and emotions are all seen as psychological processes. McLeod's framework does not explicitly address social processes in these affective dimensions, but recognizes the influence of the social context (culture) in the formation of beliefs. This is in line with the methodology of this study (which will be elaborated in chapter 3), as I primarily study the students' beliefs on an individual, psychological level and not, as such, in relation to social processes. Although, both physiological and social aspects are acknowledged through data collected from classroom observations.

Attitudes

In line with the affective domain and the concept of beliefs, *attitudes* do not have a clear and unambiguous definition. In some studies, they are only defined implicitly or rather simplistically in terms of being positive or negative, while in others they are defined more explicitly, and often with a *tripartite* definition (listing three components in attitudes) (Di Martino & Zan, 2010). Generally, these three components have been related to beliefs, behavior, and affect/emotions (e.g., Grigutsch, 1998; Hart, 1989). Thereby, beliefs are seen as a component in attitudes.

As mentioned, McLeod (1992) instead sees beliefs, emotions and attitudes as juxtaposed components of the affective domain. In his definition, attitudes are "affective responses that involve positive or negative feelings of moderate intensity and reasonable stability" (McLeod, 1992, p. 581). In McLeod's framework, attitudes can be thought of as "the end result of emotional reactions that have become automatized" (McLeod, 1989, p. 249), for example, when a student has repeated negative experiences with a certain mathematical area, the negative emotions will become stable. Hence, the emotional aspect of attitudes seems to be consistent in various definitions. In this regard, Ernest (1989) points out that attitudes often include students' liking, enjoyment and interest in math, or their opposites, as well as their self-concept and valuing of mathematics. In addition, Pehkonen and Pietilä (2003) include students' reactions to the ease or difficulty of mathematics.

In 2010, Di Martino and Zan presented a study of students' attitudes based on narratives from 1662 students across grades 1–13. Through a grounded theory approach, the analysis resulted in an identification of three main dimensions of the construct of attitude: emotional disposition (positive/negative), vision of mathematics (e.g., relational/instrumental) and perceived competence (high/low) (figure 2).

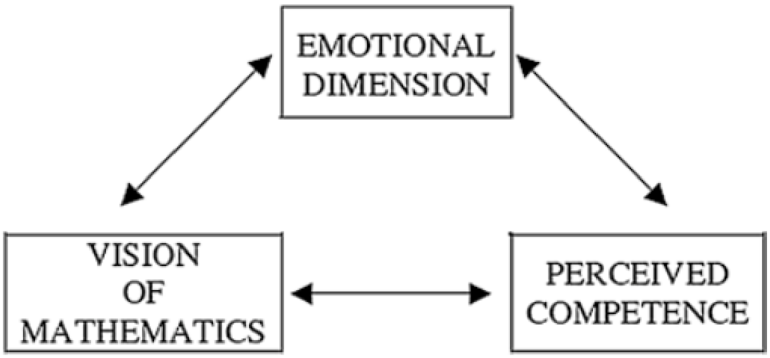


Figure 2: The three-dimensional model for attitude (Di Martino & Zan, 2010, p. 43)

Hence, what generally constitutes students' attitude toward mathematics are the feelings they have about mathematics (e.g., hate, love, fear, anger), their way of thinking about mathematics (e.g., that there are many rules, or that understanding is essential in mathematics), and their perception of how well they are doing in mathematics or how easy they find it. As the double arrows in the model illustrate, the three dimensions are interdependent. For example, a student might dislike mathematics because (s)he relates it to rules and memorization and does not feel successful in that endeavor. In that way, the study shows that a simple characterization of a student's attitude as either positive or negative is inadequate, as it only relates to the emotional dimension. The authors thus propose that an attitude can be considered negative, when at least one of the dimensions is negative (i.e., negative emotions, instrumental view of mathematics, or low competence). With McLeod's framework in mind, placing attitudes in between beliefs and emotions, Di Martino and Zan (2011) explain how their model links the three affective concepts: "The proposed model of attitude acts as a bridge between beliefs and emotions, in that it explicitly takes into account beliefs (about self and mathematics) and emotions, and also the interplay between them" (p. 479).

In the present study, the students' attitudes toward mathematics are included in the data analysis to better infer and understand their beliefs. I thereby acknowledge that attitudes are inextricably *linked* to beliefs, but not based on propositions that are assessed as true or false as suggested by Eichler and Zapata-Cardona (2016). I apply the definition of Di Martino and Zan (2010), as it not only captures the students' emotional responses, but also includes their (emotional) view of mathematics and their (emotional) perception of competence. Thus, it is possible to differentiate between attitudes and beliefs, as, for example, the statement "I like geometry" is of an emotional character with no logical value, and therefore can be categorized as an attitude. In contrast, the statement "I think it is important to learn geometry, if you for example want to be a carpenter" can be categorized as a belief, as it has a logical value and can be assessed according to its truth value (cf. Eichler & Zapata-Cardona, 2016).

Knowledge and beliefs

Another important concept in this study is *knowledge*. The relationship between knowledge and beliefs is a long-standing topic of discussion (Thompson, 1992). Furinghetti and Pehkonen (2002) suggest a distinction between objective and subjective knowledge, the former being knowledge that is generally accepted in the world of mathematics, and the latter being knowledge that is constructed by the individual—including beliefs. This distinction is also applied by Grigutsch (1998). Pehkonen and Pietilä (2003) have a similar categorization, but they include emotions in their understanding of beliefs: "Here we understand beliefs as an individual's subjective knowledge and emotions concerning objects and their relationships, and they are based usually on his personal experience." (p.1).

In contrast, beliefs and knowledge can also be perceived as distinct concepts as, for example, when Schoenfeld (1998) explains a person's behavior as a function of his/her knowledge, goals, and beliefs. Ernest (1989) describes teachers' thought structures as their knowledge, beliefs and attitudes stored as schemas in the mind. Sfard (1991) distinguishes between the formal *concept* of which there is theoretical agreement (equivalent to knowledge) and *conception*, which is a person's inner, subjective representation of a concept (which may be interpreted as equivalent to beliefs, although Sfard does not use this term). Conceptions cannot be equated with subjective knowledge, since they are only representations of the associated concept, and thereby merely reflect it. Thus, Sfard also perceives knowledge and beliefs to be separate, but mutually interdependent.

This perception—that beliefs and knowledge are closely linked—is quite common among researchers (e.g., Eichler & Erens, 2015; Op't Eynde et al., 2002; Pajares, 1992; Thompson, 1992). For example, Ernest (1989) identifies mental schemas (beliefs, knowledge and attitudes) as one of the key elements that influence teachers' practice, and Op't Eynde et al. (2002) find that beliefs and knowledge operate together in students' thinking and learning, e.g., in relation to their understanding of mathematical problems and situations. However, certain aspects are used to distinguish the two concepts. One of them is the degree of conviction (Thompson, 1992). Where beliefs can be held with various degrees of conviction, e.g., strongly or with a certain doubt, knowledge is not related to conviction. In Abelson's words, "[one] would not say that one knew a fact strongly" (Abelson, 1979, p. 360).

Another aspect is the factor of truth, which often plays an important role when discussing the relationship between beliefs and knowledge. Lester et al. (1989) state that knowledge has a truth property, as it is valid with a probability of 100%, whereas this probability value for beliefs is always less than 100% (Furinghetti & Pehkonen, 2002; Törner, 2002). As pointed out above, the truth-value of objective knowledge then needs to be publicly or officially confirmed, while the truth-value of subjective knowledge (beliefs) is assessed by the individual. Op't Eynde et al. (2002) explain how beliefs can be seen as an *individual* construction that is valued based on trustworthiness, and knowledge as a *social* construction that is valued based on truth. Eichler and Erens (2015) similarly understand beliefs as individual convictions and knowledge as inter-individual (and thereby social) convictions. Furthermore, Abelson (1979) mentions that knowledge usually excludes the self in contrast to beliefs, and Nespor (1987) points out that beliefs often include affective feelings and evaluations, which makes an objective validation impossible. For example, a person can relate a certain belief to a personal experience, or the beliefs can be associated with a judgment of good or bad. This is not the case with knowledge.

Both Abelson (1979) and Philipp (2007) suggest a differentiation between knowledge and belief. The differentiation is based on whether one can respect disagreeing positions. Where the holder of a belief usually recognizes that others might have different beliefs, there is no room for contradiction within knowledge systems. This means that what is knowledge for one person may be belief for another. For example, a student might believe that there is only one correct result to a mathematical problem, whereas another student who has had experience with quadratic equations knows differently. Moreover, an individual can consider a belief to be true—and thus consider it as knowledge—until it is proven wrong, which again complicates the distinction between beliefs and knowledge.

Where some beliefs are of a nature that means they can never become knowledge (e.g., about the origin of mathematics, what it means to be good at mathematics or one's self-efficacy beliefs), others have the potential to evolve into knowledge, which might be a goal in education. The belief of the student in the abovementioned example can become knowledge, if (s)he is presented with information and experiences that show that several correct results are possible in some situations. According to Op't Eynde et al. (2002), beliefs can become knowledge if they match socially accepted criteria, and thereby live up to the truth condition required of knowledge. As Green (1971) notes, however, this can only happen if the belief is based on evidence. This is supported by Eraut (1985) who states that personal experiences play a key role in creating knowledge, as it may function as evidence.

2.1.2. Definitions of beliefs

As already mentioned, the definition of beliefs has been debated for decades. Thirty years ago, both Pajares (1992) and Thompson (1992) requested clarification of the theoretical construct. Still, researchers operate with a variety of definitions. In the following, I present the main references for definitions and characterizations of beliefs in order to clarify the considerations behind the choice of definition employed in this study.

Rokeach (1968) presents one of the early definitions of beliefs, which in all simplicity states that: "A belief is any simple proposition, conscious or unconscious, inferred from what a person says or does, capable of being preceded by the phrase 'I believe that…'." (Rokeach, 1968, p. 113). A more domain-specific definition is presented by Schoenfeld (1985) who, in the context of mathematical problem solving, uses the term "mathematical world view": "Belief systems are one's mathematical world view, the perspective with which one approaches mathematics and mathematical tasks" (Schoenfeld, 1985, p. 45). A few years later, he adds that beliefs are "an individual's understanding and feelings that shape the ways that the individual conceptualizes and engages in mathematical behavior" (Schoenfeld, 1992, p. 358). Equivalently, Eichler et al. (2017) see beliefs in terms of perception and action: "[T]he term beliefs represents an individual's personal conviction referring to a subject that represents a disposition of the ways of receiving information and acting in a specific situation" (p. 85). With these definitions, beliefs are highly related to behavior, but they do not provide information on the interrelation of beliefs and other affective concepts. Grigutsch (1998) also uses the term "mathematical world view", but defines it differently, as he includes the concept of attitude: "… mathematical world view is a structure of attitudes which contains a wide spectrum of beliefs (cognitions), affections and intentions of actions concerning mathematics" (Grigutsch, 1998, p. 171).

Another definition is found in Richardson (1996), who synthesizes definitions from anthropology, social psychology and philosophy, thus describing beliefs as "… psychologically-held understandings, premises or propositions about the world that are felt to be true" (Richardson, 1996, p. 103). Philipp (2007) suggests that beliefs might be thought of as "lenses through which one looks when interpreting the world" (p. 258). An extensive review of literature about beliefs of both teachers and students leads him to the following, which partly builds on the one presented by Richardson:

> *Psychologically held understandings, premises, or propositions about the world that are thought to be true. Beliefs are more cognitive, are felt less intensely, and are harder to change than attitudes. Beliefs might be thought of as lenses that affect one's view of some aspect of the world or as dispositions toward action. Beliefs, unlike knowledge, may be held with varying degrees of conviction and are not consensual. Beliefs are more cognitive than emotions and attitudes. (Philipp, 2007, p. 259)*

Philipp thus applies McLeod's framework of affect in his description of the relations between beliefs, attitudes, and emotions. In this study, I employ this definition, based on the following considerations:

- The definition relates beliefs to other aspects of affect, which is in line with the framework of McLeod (1992).
- It includes considerations of stability, intensity, and cognition in the interrelation of the affective aspects, which is also coherent with McLeod's framework, as well as with structural aspects of beliefs that will be presented in section 2.1.4.
- It furthermore includes considerations of the degree of conviction and consent. This is related to the truth-value of beliefs mentioned previously (Eichler & Zapata-Cardona, 2016; Hannula, 2011)
- The definition is not exclusively oriented towards behavior or actions and may thus include other expressions and propositions of beliefs, such as thought processes, or perception of information.

2.1.3. Origin of beliefs: how are they formed, and which functions do they serve?

Richardson (1996) and McLeod (1992) point out that experiences as well as culture are major contributors to the formation of beliefs. As individuals encounter the world and gain experiences with its phenomena, they create inferences and draw conclusions that help them understand their surroundings. As these conclusions are affirmed and incorporated, they develop into beliefs (Furinghetti & Pehkonen, 2002; Pajares, 1992). Beliefs thus function as a form of selective filters that simplify the complexity of the world. Hence, beliefs often do not encapsulate the full reality. They may even work as "stabilizing knots", when knowledge is not accessible (Törner, 2014).

> *Unyielding as beliefs may be, they provide personal meaning and assist in defining relevancy. They help individuals to identify with one another and form groups and social systems. On a social and cultural level, they provide elements of structure, order, direction, and shared values. From both a*

personal and socio/cultural perspective, belief systems reduce dissonance and confusion… (Pajares, 1992, pp. 317-318)

According to Pajares (1992), beliefs form through enculturation (e.g., experiences in the form of observation or participation) and social construction (adapting others' beliefs). This is in line with Rokeach (1968), who explains that beliefs are developed from experience and reinforced by social group consensus. Also Op't Eynde et al. (2002) underline the social influence on the development of beliefs.

According to Op't Eynde et al. (2002), a person's first perception is always accepted as true, and it is only questioned or changed if it conflicts with other propositions or experiences. Once formed, the early developed beliefs become filters through which new information is perceived (Pajares, 1992). In an educational context, this means that the students' learning is highly influenced by their previous experiences, for example with mathematics (Rolka & Roesken-Winter, 2015). With time, these beliefs become robust and resistant to change even when contradictive evidence is presented, partly because they constitute the foundation for other beliefs in a belief system (cf. section 2.1.4). Hence, the earlier a belief is developed, the harder it is to alter. For example, Uusimaki and Nason (2004) found in their study on pre-service teachers' beliefs about mathematics that negative beliefs most often are formed in primary school because of negative experiences. With reference to Nisbett and Ross (1980), Pajares (1992) labels this phenomenon *the perseverance phenomena*, which will be elaborated further in section 2.4 concerning the changing of students' beliefs. It can thus be inferred that newly acquired beliefs are most vulnerable.

2.1.4. Belief systems: how are beliefs structured?

As individuals possess an immense number of beliefs, Rokeach (1968) argues that all beliefs are organized in belief systems, which are organized "in some organized psychological but not necessarily logical form". Likewise, Green (1971) states that beliefs always emerge and change as part of a belief system. Thereby, the term *belief system* is used as a metaphor for the way beliefs are organized (Green, 1971; Thompson, 1992). A belief system contains a person's "conscious and unconscious beliefs, hypotheses or expectations and their combinations" (Furinghetti & Pehkonen, 2002, p. 40) about a certain object. It is within the belief system that new experiences and information are filtered through the lenses of existing beliefs. Nespor (1987) refers to this as "episodic storage", such that beliefs are often derived from particular episodes, experiences, or events that then influence people's comprehension of subsequent experiences. Conversely, existing beliefs are also evaluated against new experiences, which might lead to modification or restructuring of the belief system (Thompson, 1992).

Concerning the properties of belief systems, Green (1971) presents three dimensions that are characteristic for the relations between beliefs within a belief system:

Dependency (Primary/derivative beliefs): Beliefs are related in a *quasi-logical* structure, which means that the order of the beliefs in a belief system is organized according to the way they are understood and connected by the individual, and not logically according to their content, as is the case in knowledge systems. Within this quasi-logical structure, there are primary and derivative beliefs, the former being the bases for the latter. Green explains a person's primary belief as a belief "for which he can give no further reason, a belief which he uses nonetheless as a reason for other beliefs" (p. 44). For example, a primary belief could be that "the ability to understand mathematics is innate", and a belief derived from this then might be "I might as well give up learning mathematics, as I do not possess the ability". However, due to the quasi-logicalness, these relations between beliefs can change. In my interpretation, one of the aims of teaching must be to modify the relations between the beliefs, so that it approaches a logical structure, thus increasing the truth-value and becoming as close to knowledge as possible.

Degree of conviction (central/peripheral beliefs): Another contrast to knowledge systems is that some beliefs are more important to a person than others. This has to do with *the psychological centrality* of the belief. Where some beliefs are central and strongly held, others are peripheral, and thus easier to change. Furinghetti and Pehkonen (2002) suggest that the degree of centrality is seen as a spectrum instead of a dual structure. As Rokeach (1968, p. 3) explains: "the more central a belief, the more it will resist change" and "the more widespread the repercussions in the rest of the system", as people strive for coherence in their belief systems (Op't Eynde et al., 2002). As an example, the above-mentioned derivative belief about mathematical ability might be of central psychological importance to the person holding it, as it functions as an explanation for low performance that frees the holder from responsibility. As peripheral beliefs are more easily changed, but not as psychologically important, it might happen that some peripheral beliefs do not "fit" the central ones. For example, some students may be influenced by an inquiry-based teaching approach in their peripheral beliefs, while having deeply rooted beliefs about mathematics as based on rules and memorization that are not compatible.

Cluster structure: Beliefs never occur independently but are always held in clusters. Thereby, it is possible to hold contradicting beliefs, as long as they belong to different clusters and are not confronted with each other, which would cause a cognitive conflict. For example, a student might believe that it is important to learn mathematics, while at the same time hold the belief that mathematics learned in school has no relevance to the real world. Schoenfeld (2015) points out that different beliefs are activated in different contexts. Giving an example: Students who concurrently state that mathematics is about rules and facts, which may be derived from experience in class, but also that understanding is important, which may be derived from the rhetoric that they have heard from the teacher. Likewise, Eichler and Erens (2015) found in their study that teachers' beliefs seem to change when referring to different mathematical domains.

In addition to these three dimensions, two other features can be said to characterize the beliefs within a belief system. One of them is their level of *consciousness*. Many of an individual's beliefs within a belief system are held at an unconscious (or implicit) level and have never been articulated (Furinghetti, 1996). If a person is not asked about his or her beliefs or presented with information that is related to them, (s)he does not necessarily think about them or become aware of them (Lester, 2002). Opportunities to articulate beliefs and thus "awaken" them by external motivation might be a way to make unconscious beliefs become conscious (Furinghetti, 1996).

Another central and influential factor, particularly in an educational context aimed at developing students' beliefs, is the way in which beliefs are held, which is emphasized by both Green (1971) and Rokeach (1968). Green differentiates beliefs that are *evidentially* held from those that are *non-evidentially* held. Evidentially held beliefs are based on rationality and are thus supported by verifiable evidence (e.g., experience), whereas non-evidentially held beliefs are derived from an external authority (e.g., a teacher, friend, parent, etc.). When beliefs are based on evidence, they can be changed with reason or further evidence. In contrast, non-evidentially held beliefs are not as likely to be changed or modified by presenting new evidence or reasonable arguments. Green embodies such a belief in the attitude: "Don't bother me with facts; I have made up my mind" (Green, 1971, p. 48).

Hence, a belief system may include primary as well as derivative beliefs, central as well as peripheral beliefs, conscious as well as unconscious beliefs, and evidentially as well as non-evidentially held beliefs. All of these dualities may thus contribute to a characterization of beliefs (e.g., in an analysis).

2.1.5. Investigating/measuring beliefs

Several researchers have mentioned the difficulties of investigating and measuring beliefs. One of the reasons that it is a complicated endeavor is the context-specific nature of beliefs (Pajares, 1992). As already mentioned, contradictive beliefs can occur in different situations and contexts.

Another reason is that beliefs cannot be directly observed and must thus be inferred from a person's statements or actions (Rokeach, 1968). Identifying the underlying beliefs of a statement or action is always based on an interpretation, which (again) might be biased by the observers' beliefs (cf. section 3.1). It is further complicated by the fact that people are not always conscious about their beliefs, or they might have personal and social reasons for stating beliefs that are different from their actual beliefs (Grootenboer & Marshman, 2016). As Schoenfeld (2015) concludes from different examples of expressed beliefs: "People lie. They lie to others, they lie to themselves. (…) Statements of beliefs do not predict actions. If you're interested in what people do, what they say they believe is of limited value" (p.

397). Likewise, Lester (2002) questions the actual validity of measuring students' beliefs, especially about the nature of mathematics, as they rarely have had any opportunities to think about this issue and thus become aware of their beliefs.

So how, then, does one proceed in the investigation and measuring of students' beliefs in a manner that is as reliable as possible? Turning to the literature, Leder (2019) finds that self-report measures (e.g., questionnaires) are widely used, which makes the consideration mentioned above highly relevant. Thompson (1992) states that research on beliefs employs qualitative methods such as interviews, classroom observations, linguistic analysis and responses to stimulation materials—often in combination, as in this study. Philipp (2007) mentions case studies and beliefs-assessment instruments as typical methods for measuring beliefs, pointing also to triangulation of data, which may contribute to a higher level of understanding of students' beliefs, as the interpretations are not only related to one expression of beliefs. The triangulation method is applied in the present study, as described in chapter 4.

As an example, Rolka and Halverscheid (2011) argue that although students' beliefs are often measured through questionnaires and interviews, younger students might struggle with reading and understanding long or complex questions, as they might not be familiar with those techniques. Hence, they combine the use of interviews with asking students to draw and write about mathematics in their study among 200 students in 5th and 6th grade. The students' beliefs are then interpreted from their drawings, using the explanative texts and in some cases an additional interview as support for, and validation of, the interpretations. Also, Picker and Berry (2000) as well as Aguilar et al. (2016) use drawings to investigate students' images of mathematicians, and Johansson and Sumpter (2010) combine Likert-scale statements, open-ended questions and a task requesting students to draw themselves doing mathematics in a questionnaire measuring primary school students' conceptions about mathematics. Likewise, McDonough and Sullivan (2014) use prompts to activate young students' beliefs. Thereby, all these studies take into account what Törner (2002) identifies as the *activation levels* of beliefs. Beliefs can have different strengths or expressions in different contexts; when measuring beliefs it is thus relevant to consider the context in which they are measured, and how expressions of them might be activated. This may, for example, be through drawings or prompts, but also certain phrasings of questions or references to certain situations may activate beliefs.

As with most studies on students' beliefs, the studies by Rolka and Halverscheid (2011) and Johansson and Sumpter (2010) are of a descriptive nature and investigate the character of the beliefs that students' hold (Roesken et al., 2011). When aiming to measure a change or development in students' beliefs, Jankvist (2015a) suggests a framework for data analysis that considers the level of *reflection* in students' beliefs, and thus does not measure the content or character of their beliefs but rather the way they hold their beliefs, cf. the notion of evidentially versus non-evidentially held beliefs presented by Green (1971). In fact, Jankvist not only considers students' beliefs about mathematics (as a discipline), but also their *views*, defined as "something less persistent than beliefs, but with the potential to develop into beliefs at a later point in time" (p. 53). The combination of beliefs and views about mathematics as a discipline is labelled *images* of mathematics as a discipline.

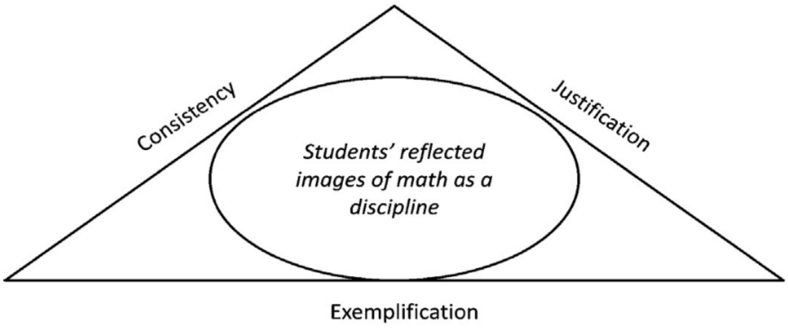


Figure 3: Students' reflected images about mathematics as a discipline as consisting of three dimensions on a basis of explicitness: consistency, justification, and exemplification (evidence). (Jankvist, 2015a, p. 54)

The framework is based on empirical findings that identify three ways that students' beliefs may develop: consistency, justification, and exemplification (figure 3). An aim for the teaching of mathematics with respect to the development of the students' beliefs may be that "the students have given thought to the images they hold, that they have reflected upon them, tried to accommodate and reformulate in case of conflicts and contradictions, and that the images they hold are evidentially held to the largest extent possible" (Jankvist, 2015a, p. 55). To determine the degree to which this aim has been obtained, the level of reflection in students' images can thus be understood by considering the three dimensions.

To my understanding, the dimension of *consistency* is related to the cluster structure in a students' belief system. As explained in section 2.1.4, this structure makes it possible for an individual to possess contradicting beliefs, as long as they are not confronted (Green, 1971). Reflection on one's beliefs and their relations may initiate such confrontations, thus forcing the individual to adjust the contradictory beliefs so that they align. A high level of consistency may thereby signal a high degree of reflection. Furthermore, if people are not conscious of their beliefs or have not yet developed beliefs on a certain issue, their statements will most likely resemble guessing, and thus not be consistent.

Exemplification is connected to the way the students' beliefs are held in relation to evidence or experiences. The more concrete examples (evidence) that a student has been presented with, and the more experiences the student has had with the belief object, the more solid the base on which the beliefs are built. Non-evidentially held beliefs are, as mentioned, not amenable to reason (Green, 1971).

Therefore, beliefs need to be based on evidence in order to be the object of critical reflection. If a person's primary beliefs are evidentially held, it will very likely influence the beliefs that are derived from it. An inference from the theoretical framework then, might be that the more beliefs are based on evidence and reflection, the closer the structure in their belief system will become to being logical instead of quasi-logical.

The dimension of *justification* is related to the students' ability to connect their beliefs to other beliefs as well as to their knowledge. Reflection on the reasons behind their beliefs can clarify which beliefs are derived from others and thus make the students aware of the relations between their beliefs. As with the dimension of consistency, the students' ability to justify their beliefs might indicate if a belief is held unconsciously, as it is unlikely that the student will be able to provide justification for something that (s)he has never previously considered.

2.2. Students' beliefs about mathematics as a discipline

As in the study by Jankvist, this study focuses specifically on students' beliefs about mathematics as a discipline. These beliefs are included in students' mathematics-related belief system that is described in section 2.2.1. Like most other concepts described in this chapter, "mathematics as a discipline" is not uniquely defined in the literature, and section 2.2.2, therefore, elaborates the term as it is applied in this study.

I then turn to the first research question of this study, which aims to investigate the character of students' beliefs to qualify the design of the intervention, as well as provide a basis for comparison in the data analysis. Here, I approach it through a systematic literature review, serving two purposes: 1) to provide information of how students' beliefs about mathematics as a discipline can be categorized and analyzed, and 2) to detect tendencies in the nature of school students' beliefs about mathematics as a discipline.

2.2.1. The mathematics-related beliefs system

A belief system is connected to a specific object (e.g., mathematics). Several researchers have presented frameworks for dimensions included in students' mathematics-related belief systems. For example, Underhill (1988) proposes four dimensions: (1) beliefs about mathematics as a discipline, (2) about learning mathematics, (3) about mathematics teaching, and (4) about oneself within a social context. With some overlap, McLeod (1992) suggests the dimensions: (1) mathematics, (2) self, (3) mathematics teaching, and (4) social context. In the framework by Kloosterman (1996), only two dimensions are presented: (1) beliefs about mathematics, and (2) beliefs about learning mathematics, though the second has three sub-dimensions: beliefs about oneself as a learner, about the role of a teacher, and other beliefs about learning mathematics. Some of the dimensions are repeated in the framework presented by Pehkonen (1995), who includes (1) beliefs about mathematics (including, the nature of mathematics, the subject of mathematics and the origins of mathematical tasks), (2) about oneself within mathematics, (3) about mathematics teaching, and (4) about mathematics learning.

As a reaction to the lack of consensus, Op't Eynde et al. (2002) synthesize these four categorizations, presenting their own framework for students' mathematics-related beliefs. The framework consists of three dimensions (figure 3):

A framework of students' mathematics-related beliefs
1. Beliefs about mathematics education a) beliefs about mathematics as a subject b) beliefs about mathematical learning and problem solving c) beliefs about mathematics teaching in general
2. Beliefs about the self a) self-efficacy beliefs b) control beliefs c) task-value beliefs d) goal-orientation beliefs
3. Beliefs about the social context a) beliefs about the social norms in their own class – the role and the functioning of the teacher – the role and the functioning of the students b) beliefs about socio-mathematical norms in their own class

Figure 4: A framework for students' mathematics-related beliefs (Op't Eynde et al., 2002, p. 28)

What might be surprising is that Op't Eynde et al. (2002) do not include beliefs about mathematics as a discipline, but confine to dimensions related to mathematics *education* or mathematics as a *school subject.* The authors explain that:

> *[...] students' beliefs about math education reflect their view on what mathematics is like: the perspective with which they approach mathematics and mathematical problems and tasks. [...] Furthermore, what we fundamentally think about mathematics and mathematical knowledge is closely related to what we think mathematics learning, on the one hand, and mathematics teaching, on the other, are like (Hofer & Pintrich, 1997). Therefore, we consider these three kinds of beliefs to be closely related. Clustered, they seem to constitute three interrelated subsets of beliefs about mathematics. (Op't Eynde et al., 2002, p. 29).*

One might argue, though, that this explanation is still set within an educational context and does not explicitly consider the aspects of mathematics that are connected to the world outside school or to mathematics as a science—except, of course, to the extent that these aspects are included in the school subject. As pointed out in the introduction of this dissertation, an often stated problem in mathematics education research is that students are missing a feeling of relevance and meaning in relation to the subject of mathematics, and that they often see no relation between the mathematics they learn in school and the real world (e.g., Schoenfeld, 1992). Therefore, it might be appropriate to reconsider including non-school aspects of mathematics in mathematics teaching. Students' beliefs about the nature of mathematics can have a huge influence on their relationship with the subject (Kloosterman, 2002). Hence, Jankvist (2015a) argues for the addition of a fourth dimension concerning beliefs about mathematics *as a discipline,* which includes aspects of, but is not restricted to, the *scientific* discipline of mathematics. This addition to the belief system is partly based on the before-mentioned considerations about how certain learning goals, for example connected to developing students' democratic and critical competencies, involve a development of the students' beliefs about mathematics in a societal, cultural, and scientific context. Therefore, it is necessary to include the development of students' beliefs about mathematics as a discipline as a goal in a democratic society.

These beliefs are developed in the context of mathematics education and, therefore, in interplay with the three other dimensions of Op't Eynde et al. (2002). Hence, instead of placing the fourth dimension in the same plane as the original ones, making the model a square, the "mathematics as a discipline" is placed "outside" the original triangle, which then forms the "base" of a tetrahedron (figure 4).

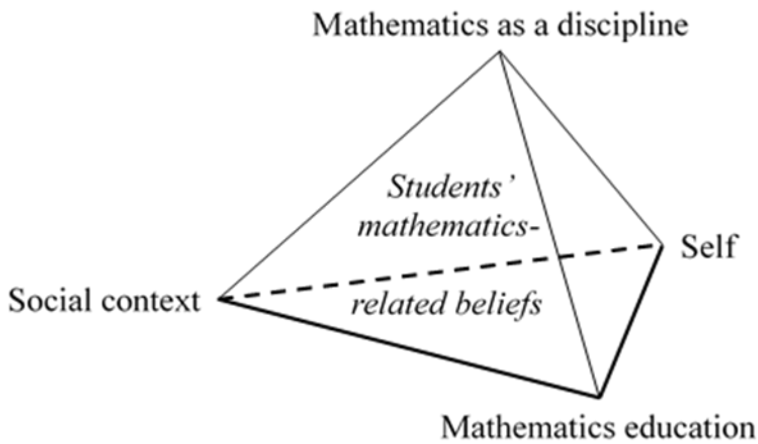


Figure 5: Jankvist's expansion of "Constitutive dimensions of students' mathematics-related belief systems" (Jankvist, 2015a, p. 45) The bottom triangle constitutes the original model by Op't Eynde et al. (2002).

2.2.2. Mathematics as a discipline

According to Jankvist (2015a, p. 45), the fourth belief dimension includes "beliefs concerning mathematics as a pure science, an applied science, a system of tools for societal practices, as well as the philosophical and epistemological nature of mathematical concepts, theories etc." As mentioned in the introduction, mathematics as a *discipline* is here understood as the nature of mathematics as a scientific as well as an applied discipline, including the issues concerning the ontology, epistemology, methodology, and application of mathematics. It is thus not restricted to mathematical knowledge or content, but also involves philosophical and historical aspects of mathematics, as well as considerations concerning the role of mathematics in the world. Mathematics as a discipline may be said to concern all that might be involved—internally and externally—in the science and use of mathematics, and in the job of a mathematician.

What is essential in this context is that there are different ways of perceiving the discipline of mathematics. An often applied categorization is the one by Ernest (1989), who identifies three philosophies of mathematics among teachers (although this framework has since been found suitable for categorizing students' beliefs as well): the *instrumentalist* view, where mathematics is seen as "a set of unrelated but utilitarian rules and facts"; the *Platonist* view, which characterizes mathematics as "a static but unified body of certain knowledge" that is discovered and not invented; and the *problem-solving* view, with a perception of mathematics as "a dynamic, continually expanding field of human creation and invention" (Ernest, 1989, p. 249). This three-part categorization is also found in Dionne (1984), who distinguishes between a *traditionalist*, a *formalist* and a *constructivist* view of mathematics, presented in Törner (1998) as the *toolbox*, *system*, and *process* view. Parallel to the instrumentalist view, the toolbox view is characterized as perceiving mathematics as a set of rules and procedures, and mathematical activity as calculating and using rules, procedures, and formulae. With the system view, mathematics is seen as based on rigorous proofs, definitions, axioms, and a rigorous mathematical language, and the process view emphasizes the role of the relations between different notions and statements (Furinghetti & Morselli, 2011).

However, these are only examples of how "mathematics as discipline" or similar notions are described, defined, or categorized in mathematics education research. In the following section, I present the results of a qualitative, systematic literature review (Grant & Booth, 2009) to address this matter, along with the question of which beliefs students actually possess about mathematics as a discipline when studied empirically.

2.2.3. Literature review on students' beliefs about mathematics as a discipline

Certain criteria have been taken into account in the search process, ensuring a systematic selection of relevant literature:

A. The review only includes studies concerning students in primary and secondary school. Students on a higher level will often have chosen a certain educational path and, thereby, have a bias regarding their interest in mathematics.

B. Despite the wide representation of studies about students' beliefs about mathematics in general, this review is restricted to addressing students' beliefs about the nature of mathematics, or mathematics as a discipline. However, some studies investigating students' beliefs about "what is mathematics?" are included, as they cover the essence of beliefs about mathematics as a discipline. The ambiguity of the term "mathematics as a discipline" is reflected in its various characterizations, including the content. Therefore, studies addressing only parts of mathematics as a discipline (e.g., problem solving, the history of mathematics, or the role of mathematics in society) have been excluded, as such studies might exclude aspects that other studies consider part of mathematics as a discipline. Moreover, this review investigates students' beliefs about mathematics as a discipline as an overarching concept, not the individual content parts or perspectives on this.

C. Only literature published within the last 20 years has been included in the search process, following the assumption that earlier relevant and important literature will be cited in studies that are more recent. Hence, references appearing to be relevant, and references with several citations in the selected studies have also been included, if they met the inclusion criteria.

D. Only peer-reviewed studies have been included to ensure the validity of the selected literature. Both quantitative and qualitative studies have been included in the search process.

E. As described in section 2.1.1, the concept of beliefs is somewhat unambiguously defined. Even though 'beliefs' is the most commonly used term in the resulting studies (15 studies), synonyms are also represented: conceptions (in 2 studies), views (in 3 studies), and images (in 1 study). In the screening process, it has been assessed whether these terms cover aspects relevant to this study. In this review they are thus considered as similar, if not identical, and quite closely related notions that in essence cover the same phenomenon.

Williams and Leatham (2017, p. 377) identify the twenty most important journals in mathematics education. To cover these, the search for literature was conducted in two databases: ERIC[2] and in Web of Science[3]. Furthermore, manual searches have been conducted in proceedings from the MAVI 16-35, PME 24-43 and CERME 2-11 conferences. The search in databases resulted in 292 studies, which were imported into the Covidence software. Seven duplicates were removed. The remaining 285 studies were screened against title and abstract following criteria A through E. This led to further exclusion of 275 studies. Five studies were imported from CERME proceedings, 10 studies from MAVI proceedings, and 8 studies from PME proceedings. In total, 33 studies were full-text screened, resulting in an exclusion of 26 studies, 5 of these related to criterion A (wrong sample group), 20 to criterion B (not mathematics as a discipline), and 1 to criterion B (wrong aspect of affect). Finally, 7 studies were included in this review, and from their references, I included another 12 more studies through a so-called 'snowballing process' (i.e., citation tracking in reference lists). Hence, the review is based on 19 studies in total.

<u>Step 1:</u>	292 references imported from databases for screening
	7 duplicates removed (285 remaining)
<u>Step 2:</u>	285 studies screened against title and abstract
	275 studies removed (10 remaining)
<u>Step 3:</u>	23 studies imported from conference proceedings for full-text assessment
<u>Step 4:</u>	23 + 10 = 33 studies assessed for full-text eligibility
	26 studies excluded: 5 (criterion A); 20 (criterion B); 1 (criterion E). (7 remaining)
<u>Step 5:</u>	11 studies included from snowballing
<u>Step 6:</u>	In all 19 studies included

Figure 6: Overview of review process and studies excluded based on inclusion criteria. Exclusions related to criteria C and D tool place in step 1.

The frameworks and findings in the included studies have been analyzed from three perspectives:

a) The characterization of beliefs about mathematics as a discipline. This perspective is found in 10 studies. Some researchers apply existing frameworks or categories to their data, whereas others develop their own framework. This perspective only includes studies with a clear categorization, definition, or framework.

[2] Search string in ERIC: noft((belief* OR view* OR perception* OR conception* OR image* OR understanding*) AND math* AND (discipline OR nature) AND (student* OR pupil* OR child*) AND (school OR primary OR secondary) NOT ("STEM" OR teacher*)) AND la.exact("English") AND PEER(yes) AND pd(>20001231) (April 22, 2021)

[3] Seach string in Web of Science TS=((belief* OR view* OR perception* OR conception* OR image* OR understanding*) AND math* AND (discipline OR nature) AND (student* OR pupil* OR child*) AND (school OR primary OR secondary) NOT (STEM OR teacher*)) (April 22, 2021)

b) Empirical findings of the type of beliefs that students actually possess were found in 14 of the studies.
c) Eight of the included studies address the value of students' beliefs, strongly indicating that certain beliefs about mathematics are preferable to others, as they are considered "appropriate", "favorable" or "ideal", and should thus be pursued in the students' learning and their cognitive development. Consequently, there are also beliefs that are considered undesired or unfavorable, generally because they do not support the students' learning, critical sense, motivation, etc.

The distribution of the studies within the three perspectives is listed below (table 1).

Table 1: Distribution of studies according to analysis perspectives

Characterization of beliefs	**Empirical findings**	**Quality/value of beliefs**
Borasi (1993) Grigutsch (1998) Gattermann et al. (2012) Grady (2018) Grevholm (2011) Grouws (1996) Halverscheid & Rolka (2006) Jankvist (2015a) Rolka & Halverscheid (2011) Schoenfeld (1992)	Garofalo (1989) Gattermann et al. (2012) Grevholm (2011) Grigutsch (1996) Grootenboer (2003) Grouws (1996) Halverscheid & Rolka (2006) Kloosterman (1996, 2002) McDonough (1998) Rolka & Halverscheid (2011) Schoenfeld (1989, 1992) Underhill, 1988	Borasi (1993) Schoenfeld (1992) Gattermann et al. (2012) Grouws (1996) Grigutsch (1998) Spangler (1992) Jankvist (2015a) Ernest (2015)

Characterization of beliefs

As mentioned earlier, there are various ways of characterizing what actually constitutes and is included in students' beliefs about mathematics as a discipline. One way is to define the *content* of these beliefs in the form of a set of categories or issues. This is done in three of the studies included in this review.

Content of beliefs about mathematics as a discipline

Borasi (1993) divides beliefs about mathematics into four categories, based on existing research: two related to mathematical *activity* (nature and scope), and two related to mathematical *knowledge* (nature and origin). Similar categories can be found in the framework of Grouws (1996), which is used to analyze students' conceptions of mathematics, although with other definitions of the dimensions of mathematical knowledge (*composition, structure and status*) and mathematical activity (*doing* mathematics and *validating* ideas in mathematics). Moreover, Grouws adds two dimensions: the *usefulness* of mathematics and the *learning* of mathematics.

With the latter addition, Grouws inserts the discipline of mathematics into an educational context. However, as described in section 2.2.1, Jankvist (2015a) argues for adding the dimension of mathematics as a discipline to Op't Eynde et al.'s (2002) model of students' mathematics-related beliefs, to include issues related to non-school settings. Inspired by Spangler (1992), Jankvist characterizes this dimension of the belief system through a set of questions:

> *[H]ow, when and why mathematics came into being; if mathematics is discovered or invented; where mathematics is applied; if it has greater or lesser impact on society today than previously; if mathematics can become obsolete; what mathematicians do; if mathematics is a scientific discipline. (Jankvist, 2015a, p. 45)*

<u>A spectrum of beliefs</u>

Eight of the included studies characterize students' beliefs about mathematics as a discipline by characterizing an individual's *perception* of, or *perspective* on, mathematics (often within a spectrum), exemplified by the frameworks by Ernest (1989), Dionne (1984), and Törner (1998). The majority of these studies present somewhat similar spectra. In one end of these spectra, mathematics is overall perceived as a static, rigid, and rule-based discipline, and in the other end, it is seen as a relativistic, dynamic, and applicable "science of patterns", as described by Schoenfeld:

> *At one end of the spectrum, mathematical knowledge is seen as a body of facts and procedures dealing with quantities, magnitudes, and forms, and the relationships among them; knowing mathematics is seen as having mastered these facts and procedures. At the other end of the spectrum, mathematics is conceptualized as the "science of patterns," an (almost) empirical discipline closely akin to the sciences in its emphasis on pattern-seeking on the basis of empirical evidence. (Schoenfeld, 1992, p. 334).*

Dualistic	< ------------------------ >	**Relativistic**
	1. composition of mathematical knowledge	
facts, formulas and algorithms	< ------------------------ >	concepts, principles and generalizations
	2. structure of mathematical knowledge	
collection of isolated pieces	< ----------------------- >	coherent system
	3. status of mathematical knowledge	
static entity	< ------------------------ >	dynamic field
	4. doing mathematics	
results	< ------------------------ >	sense-making
	5. validating ideas in mathematics	
outside authority	< ------------------------ >	logical thought
	6. essence of learning mathematics	
memorizing	< ------------------------ >	constructing and understanding
	7. usefulness of mathematics	
school subject with little value in life	< ------------------------ >	useful endeavor

Figure 7: Dimensions for the conceptions of mathematics and their poles on a range from dualistic to relativistic (my extraction from Grouws, 1996).

Based on the framework of Oaks (1989), Borasi (1993) and Grouws (1996) characterize mathematics-related beliefs on such a spectrum, ranging from dualistic to relativistic. The seven dimensions which Grouws, as mentioned above, uses to categorize beliefs about mathematics as a discipline are each described as a two-poled continuum (figure 7), showing the extremes of the spectrum.

Another example of a similar spectrum is presented by Grigutsch (1998) in his study of the development of secondary school students' views of mathematics. Here, two contrasting poles are identified: a schema-orientation (aspects S and RS below) and a process/application-orientation (aspects P and A). These two poles (in essence) resemble the dualistic and relativistic perspectives, yet the spectrum between them is characterized using five different aspects, which enables a more detailed analysis of students' beliefs (Grigutsch, 1998, pp. 174-176):

F: The Formalism-Aspect (mathematics as logical and precise thinking)
P: The Process-Aspect (mathematics as a method for considering, understanding, and solving problems)
A: The Application-Aspect (mathematics as useful in daily life)
S: The Schema-Aspect (mathematics as a collection of rules and procedures)
RS: The Rigid Schema-Orientation (mathematics is only learned (memorized) to pass exams).

Still, it could be argued that the framework does not in fact represent a spectrum but simply five different aspects, as the Formalism-Aspect does not necessarily fit into the dualistic/relativistic spectrum, but to a higher degree reflects what Ernest identifies as the Platonist view (cf. section 2.2.1).

In their study of students' epistemological beliefs in mathematics, Gattermann et al. (2012) distinguish between naïve and sophisticated beliefs, relating the latter to deep-processing learning. Using items from existing large-scale assessment tools such as PISA and TIMSS, a questionnaire is used to measure the students' beliefs. These are then classified within six conceptual aspects, which to a large extent resemble those of Grigutsch (1998). Three of these aspects are identified as naïve epistemological beliefs with a perception of mathematics similar to the dualistic view: (1) rigid schemes ("exercises in mathematics always have only one right solution"), (2) schematic conception (mathematics as a collection of calculation methods and rules), and (3) realistic conception ("all mathematical problems have already been solved"). Likewise, the aspects identified as sophisticated epistemological beliefs are similar to the relativistic view: (4) relativistic (mathematics as a coherent system), (5) processes (mathematics can be discovered and constructed by oneself) and (6) relevance/application (relevant for everyday life).

Several of the studies (Gattermann et al., 2012; Grady, 2018; Grevholm, 2011; Halverscheid & Rolka, 2006; Rolka & Halverscheid, 2011) rely on the before-mentioned three-part categorizations of beliefs (or views) by Ernest (1989) or Dionne (1984) (cf. section 2.2.2). As was the case with the framework by Grigutsch (1998), these categorizations are not necessarily defined within a spectrum, but rather as three different perspectives on mathematics that are not opposites, as they can be simultaneously present in a person's belief system, depending on the context and the belief object. Yet, Ernest (1989) does place the three perspectives in a hierarchy, with the instrumentalist view on the lowest level, and the problem-solving view at the highest level.

Grady (2018) introduces an alternative approach to students' beliefs. Her study presents a framework for characterizing and analyzing students' enacted conceptions of the nature of mathematics, by investigating their *behavior* instead of the often-used self-reporting. Using behavioral indicators, the framework assesses the degree to which the students view mathematics as *sensible*, defined as perceiving mathematics as "a coherent, connected system that can be reasoned about and used to describe and reason about the world at large" (Grady, 2018, p. 127). Thereby, this definition bears resemblance with both Ernest's problem-solving view, Grigutsch's process/application-orientation, and Oaks' relativistic view. The students' view of mathematics as sensible is assessed on the basis of four behavioral categories (besides actually stating that mathematics makes sense): (1) strategizing (e.g., seeking alternative solutions or discussing methods), (2) expecting explanations (e.g., inquiring, reasoning, and justifying), (3) expecting/seeking connections (within mathematics and in other contexts), and (4) assuming authority (e.g., formulating problems of their own or testing an answer using an alternative strategy). Thereby, the framework contributes to an understanding of the action-oriented aspects of students' beliefs about mathematics as a discipline.

Following this presentation of frameworks for characterizing students' beliefs about mathematics as a discipline, I turn to a study of what beliefs students' actually hold when investigated empirically, in the form of concrete examples of students' beliefs, as well as in relation to the presented frameworks.

Students' actual beliefs about mathematics as a discipline

Based on existing research (Schoenfeld, 1992) as well as experience and discussions with teachers and students (Garofalo, 1989), the studies in this review present concrete examples of beliefs typically held by students. One of these beliefs is that a mathematical problem has only one correct solution and that it can only be solved by applying the correct rule, procedure, or formula, usually the one most recently shown by the teacher. With such a belief, mathematics is most likely perceived as a disintegrated set of rules and formulas to be memorized and applied appropriately. Also, it seems to be related to another typical belief about the nature of mathematics, namely that it must be transferred (normally from the teacher or the textbook to the student) and memorized, as "normal" people are not capable of producing it. Thereby, students feel unable to create mathematics on their own (Garofalo, 1989), thus making a deeper understanding of the rules and formulas irrelevant—which is also the case for formal proof (Schoenfeld, 1992). Another typical belief concerns the role of mathematics in the real world, where it is believed to have little relevance, as it is purely seen as a school subject (ibid.). These beliefs generally reflect the above-mentioned *dualistic* perspective. Underhill (1988) supports this in the review of mathematics learners' beliefs. Overall, students seem to emphasize algorithms and memorization as central in mathematics, promoting what Skemp (1976) classifies as instrumental understanding (merely being able to apply a procedure without knowing why it works or what it means), contrary to relational understanding (knowing both what to do and why).

A large part of the studies included in this review present empirical findings that confirm this tendency. In their study of 28 5th grade students, Halverscheid and Rolka (2006) found that half of them possess an instrumentalist view of mathematics. Five years later, the authors reported their result from another study (Rolka & Halverscheid, 2011) of 200 5th and 6th grade students' mathematical world view (cf. section 2.1.6). After asking the students to make a drawing that depicted what mathematics meant to them, they found that although most students had mixed mathematical world views, there was a clear dominance of the instrumentalist view (figure 8).

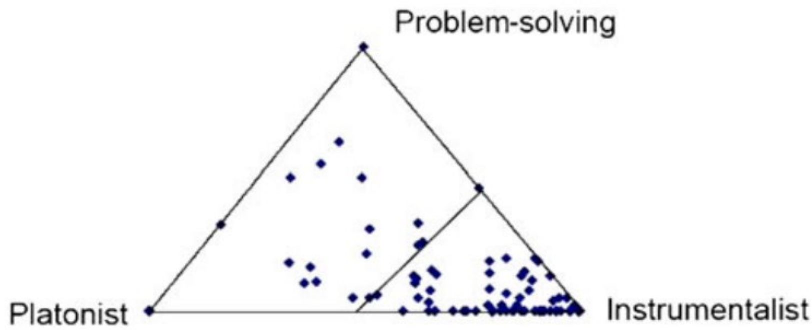


Figure 8: Students' mathematical world view. Weights in barycentric co-ordinates (Rolka & Halverscheid, 2011, p. 531)

Both Grevholm (2011), Grootenboer (2003), and Kloosterman (1996, 2002) found that students' beliefs about mathematics as a discipline are generally connected and perhaps even limited to numbers and calculations. However, most students do not seem to have given much thought to aspects related to mathematics as a discipline (Kloosterman, 2002, Grevholm, 2011).

Hence, the selected literature largely points to students' beliefs as being rather dualistic or instrumental. Yet, a few of the empirical studies present a slightly more complex characterization. In an in-depth study of two 3rd grade students' engagement in mathematical procedures, McDonough (1998) found their beliefs to be more multifaceted, broad and subtle than what is suggested in existing research. During a period of five months, the two students were interviewed about the nature of mathematics in ten one-to-one interviews, for example, by discussing personal definitions for mathematics, asking them to finish the sentence "Math is like…", and looking at photographs picturing both school and non-school activities. In both cases, the students' beliefs first appeared to be quite simple to classify, but during the ten interviews, the complexity and ambiguity increased. For example, one of the students initially appeared to focus on numbers, but the subsequent discussions revealed that measurement and estimating was more significant for her, and she primarily associated mathematics with non-school activities.

In the aforementioned study by Gattermann et al. (2012), the average score of the 145 participating secondary school students was relatively high regarding the *sophisticated* epistemological beliefs. Conversely, the average score related to *naïve* beliefs was low. However, when measuring the aspects of *relativistic* and *schematic* conceptions of mathematics, the contrary appeared to apply, as the average score on the former was low, and the latter was high. It thus seems that even though the students viewed mathematics as process-oriented and useful in everyday life, they did not perceive it as a coherent system, but rather as a collection of precise procedures and rules for calculation.

Such apparently contradictory beliefs were also found in the study by Schoenfeld (1989) among 230 mathematics students in grades 10-12. Responding to a questionnaire about their mathematics-related beliefs, including their view of mathematics as a discipline, the students, for example, indicated that creativity, logic, and discovery are characteristics of mathematics, while they simultaneously accentuated the centrality of memorization when learning mathematics. The students generally distinguished abstract mathematics and school mathematics. Schoenfeld argues that the reason might be that the students' behavior is determined by their *experiences* with mathematics, rather than what they perceive as "appropriate" beliefs. Schoenfeld (1992) as well as Garofalo (1989) and Grootenboer (2003) emphasize that students' beliefs to a large extent are formed on the basis of their experiences in the classroom. These beliefs thus reflect how mathematics is presented, enacted, and evaluated in the educational system.

Grouws (1996) presents a rather noteworthy result in his comparative study among 112 average and 55 talented high school students. Where both groups generally perceived mathematics as a dynamic and useful field, their conceptions of doing and learning mathematics were remarkably different. While the average students—parallel to the findings of Gattermann et al. (2012)—viewed mathematics from a dualistic perspective, perceiving it as a discrete system of procedures and rules, based on memorization, the talented students tended to take a more relativistic perspective on mathematics. They perceived it as "a field composed of a system of coherent and interrelated concepts and principles, which is continuously growing. Doing mathematics is a sense-making process in which one must rely on personal thought and reflection to establish the validity of that knowledge" (Grouws, 1996, p. 31). Grigutsch

(1998) found an equivalent result in the above-mentioned study, as there was a larger representation of the process/application-oriented view of mathematics among the 12th grade students in the high-performance class, compared to those in the basic level class, where the schema-orientation seemed to be prominent. The two poles of the spectrum appeared to become distinct with age, as the beliefs among the participating students in grade 6 and 9 represented a mix of the five different aspects in Grigutsch's framework described above. These two studies, thereby, point to certain beliefs that might be connected to high performance in mathematics, thus making them preferable and pursuable. The following section address this matter.

Are some beliefs better than others?

When it comes to valuing which beliefs might be unfavorable for learning and which might be preferred, there seems to be consensus among the studies in this review. In contrast to beliefs belonging to the relativistic end of the spectrum, a dualistic perspective on mathematics is not considered conducive for students' learning, self-concept, or motivation. As pointed out by Borasi (1993) and Schoenfeld (1992), dualistic beliefs impoverish mathematics and do not reflect its nature, whereas beliefs in the relativistic end of the spectrum are perceived as unambiguously beneficial. This is for example the case in Gattermann et al. (2012), where sophisticated epistemological beliefs were found to be connected to a higher degree of self-concept and performance than naïve beliefs. Likewise, the relativistic perspective seemed to be prevalent among talented students in the aforementioned study by Grouws (1996), which corresponds with the result of Grigutsch (1998), where motivation, high performance, and a positive self-concept in mathematics were linked to the process/application-orientation.

Aiming at simultaneously assessing students' beliefs about mathematics and making the students' aware of these, Spangler (1992) posed 11 open-ended questions, which indirectly imply that beliefs belonging to the relativistic end of the spectrum are preferred. The students were, for example, asked to reflect on the possibility that mathematics exceeds memorization or computation, that a mathematical problem may entail different but equally accurate answers, and that mathematics is applied in numerous non-school situations, fields, and careers. Also, the abovementioned framework by Grady (2018) indicates that a behavior, which reflects a relativistic perspective on mathematics, is desired.

Jankvist (2015a) suggests a different way of identifying favorable beliefs about mathematics as a discipline. Instead of valuing beliefs according to their content, the ideal beliefs are seen as those held evidentially (cf. section 2.1.4), i.e., based on evidence from experience, reasoning, examples etc. As is already established, such beliefs are more likely to be changed with reason or through reflection. Hence, mathematics education should aim for providing opportunities for experiences and reflection on which the students may build nuanced beliefs about mathematics as a discipline. In a didactical perspective, Jankvist (2015a) links the development of students' beliefs about mathematics as a discipline to the before-mentioned three forms of mathematical overview and judgment regarding the application, historical development and nature of mathematics (Niss & Højgaard, 2011, 2019), which will be elaborated in section 2.3. These may be perceived as somewhat equivalent to one of the visionary aims for school mathematics presented by Ernest (2015). The intention of these aims is to "contribute to students' mathematical confidence, mathematical creativity, social empowerment and broader appreciation of mathematics" (Ernest, 2015, p. 189). Particularly the latter (broader appreciation of mathematics) is connected to the students' beliefs about mathematics as a discipline, involving an increased awareness of the aspects listed below (p. 191-192):

- *(…) mathematics as a central element of culture, art and life, present and past, which permeates and underpins science, technology and all aspects of human culture.*
- *(…) the historical development of mathematics, the social contexts of the origins of mathematical concepts, its symbolism, theories and problems.*
- *(…) mathematics as a unique discipline, with its central branches and concepts, as well as their interconnections, interdependencies, and the overall unity of mathematics.*
- *(…) the way mathematical knowledge is established and validated through proof […], as well the limitations of proof.*
- *(…) a qualitative and intuitive understanding of many of the big ideas of mathematics (pattern, symmetry, structure, proof etc.)*

While the first and second of these aspects relate to the first and the second type of overview and judgment (the application and the historical development of mathematics, respectively), the last three aspects are all connected to the third type that concerns the nature of mathematics as a subject area.

Concluding remarks

There seems to be a clear pattern in the research on students' beliefs about mathematics as a discipline. Firstly, most of the studies in this review relate this dimension to a non-school setting, including aspects of mathematics related to the "real world" and to the scientific discipline. The classification of students' beliefs generally rely on a more or less detailed form of the dualistic/relativistic spectrum, ranging from a view of mathematics as a static body of rules and procedures to be memorized, to a dynamic, coherent, sensible system, with an essential place in the world and in life. Secondly, the empirical findings of the studies indicate that students generally tend to view mathematics from a dualistic perspective, emphasizing numbers, memorization, and computations.

Several researchers stress that these beliefs reflect the students' experiences in the classroom. Hence, it is essential to consider and pay attention to which beliefs are favorable in regard to students' learning, motivation, and enjoyment of mathematics. In this regard, the literature provides several examples of the benefits of a relativistic perspective on mathematics. Conclusively, a substantial adjustment of students' beliefs about mathematics as a discipline must be recommended, so that they become held evidentially, based on evidence and reflection as well as on experiences, which represent the relativistic perspective on mathematics.

What is further remarkable is the relatively low number of studies concerning primary and secondary school students' beliefs about mathematics as a discipline located in the search process. This might be connected to the search *strategy*, which included rather narrow criteria, for example, in relation to age group, time span, or object. Perhaps an expansion on some of these areas (e.g., including studies concerning mathematics in general or addressing the individual issues of mathematics as a discipline) would result in a higher number of relevant hits. Yet, the apparently low interest in the field must be taken into account, not least in light of the significance students' beliefs have to their learning and motivation, as well as the disparity between the beliefs that are considered favorable and those which they actually possess.

In the following section, the notion of mathematical overview and judgment is elaborated. As mentioned, this notion can be related to the development of students' beliefs about mathematics as a discipline (Jankvist, 2015a), and thus contribute to a teaching approach that include considerations of how these beliefs can be developed.

2.3. Overview and judgment

Mathematical overview and judgment (OJ) is part of an influential Danish framework concerning mathematical competencies. I begin this section by introducing this framework and the ideas and intentions behind it. I then turn to a thorough elaboration of the three forms of overview and judgment, followed by considerations of the relationships between this notion and the belief dimension about mathematics as a discipline. Finally, I explain how mathematical overview and judgment is included in the Danish mathematics curriculum for primary and lower secondary grades (grades 1-9).

2.3.1. The Danish competencies framework

In 2002, the first Danish edition of the so-called *KOM-report* "Competencies and Mathematical Learning" by Niss and Jensen (2002, English translation: Niss & Højgaard, 2011) was published. The report was originally initiated by the Danish Ministry of Education, partly to develop the mathematics education program in Denmark on all levels, creating progression and coherence through the entire educational system, and partly to dispute the so-called *syllabusitis*, where mastering of a subject is identified with the proficiencies related to its syllabus (Lewis, 1972).

Since then, the KOM-report's described competencies framework has had a significant impact in the Danish mathematics program on all educational levels. It is based primarily on Niss' own ideas, experience, and substantial knowledge within the field of mathematics and mathematics education, as well as on extensive discussions with peers of what it means to 'master mathematics'. The report defines overall mathematical competence as:

> *[H]aving knowledge of, understanding, doing, using and having an opinion about mathematics and mathematical activity in a variety of contexts where mathematics plays or can play a role. (Niss & Højgaard, 2011, p. 49)*

It can thereby be defined as a mathematical *meta*-competence, as it is described as relying on two pillars: eight action-oriented competencies[4], and three forms of overview and judgment.

The eight mathematical competencies are:

a) Mathematical thinking competency
b) Problem handling competency
c) Modeling competency
d) Reasoning competency
e) Representation competency
f) Symbols and formalism competency
g) Communication competency
h) Aids and tools competency

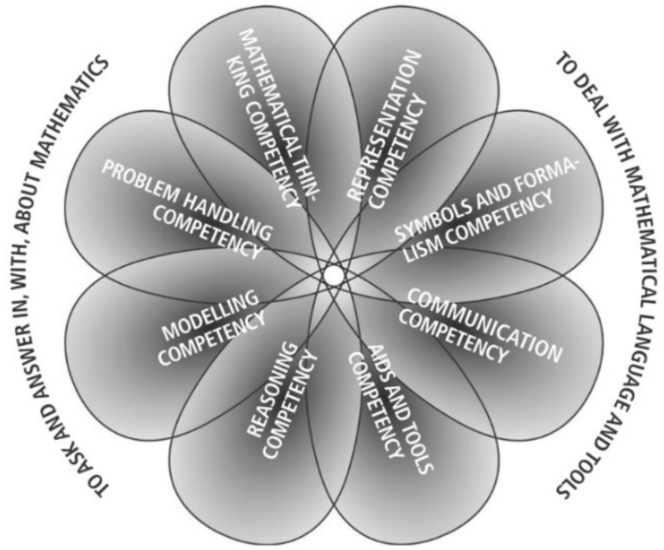


Figure 9: A visual representation of the mathematical competencies (Niss & Højgaard, 2019, p. 19)

A mathematical competency is a "well-informed readiness to act appropriately in situations involving a certain type of mathematical challenge" (Niss & Højgaard, 2011, p. 49). So, where the overall "mathematical competence" concerns the ability to act in a variety of contexts, a "mathematical competency" addresses specific challenges.

[4] The difference between *competence* and *competency* is intentional and held to distinguish the overall concept of mathematical competence from the action-oriented competencies.

In contrast, the three forms of OJ are not behavioral, but aim to develop insight into "essential features of mathematics as a discipline" (Niss & Højgaard, 2019, p. 24). They are defined as a *set of views* regarding "the relations between mathematics and conditions and chances in nature, society and culture" (Niss & Højgaard, 2011, p. 74). Mathematical OJ is based on knowledge as well as views (or beliefs) about mathematics as a discipline and it enables an individual to make appropriate and meaningful choices in mathematics-related situations. Thereby, the mastery of mathematics requires a combination of mathematical competencies, as well as overview and judgment. The possession of knowledge and beliefs about the nature of mathematics does not in itself make one mathematically competent. On the other hand, it is necessary to draw on an overall knowledge and idea to be able to act appropriately in mathematical situations. However, overview and judgment is not limited to qualifying actions:

> *Having an overview and being able to exercise judgement are of significant importance for the creation of a balanced picture of mathematics, even though this is not behavioural in any simplistic way. The point is that the object of this judgement is mathematics as a whole and not specific mathematical situations or problems. (Niss & Højgaard, 2011, p. 74)*

The three forms of overview and judgment (OJ) concern (Niss & Højgaard, 2011, p. 74; 2019, pp. 24-25)

OJ1. The actual application of mathematics within other disciplines and fields of practice.
OJ2. The historical development of mathematics, seen from internal as well as from socio-cultural perspectives.
OJ3. the nature of mathematics as a subject area.

Their essence is described by the authors of the KOM-report through examples of questions that are related to each aspect. The first form of overview and judgment concerning the *actual application of mathematics* (OJ1) are thus characterized by questions such as (Niss & Højgaard, 2011, p. 75):

- *"Who, outside mathematics itself, actually uses it for anything?"*
- *"What for?"*
- *"Why?"*
- *"How?"*
- *"By what means?"*
- *"On what conditions?"*
- *"With what consequences?"*
- *"What is required to be able to use it?" Etc.*

Examples of questions connected to the second form of overview and judgment concerning the *historical development of mathematics* (OJ2) are (p. 76):

- *"How has mathematics developed through the ages?"*
- *"What were the internal and external forces and motives for development?"*
- *"What types of actors were involved in the development?"*
- *"In which social situations did it take place?"*
- *"What has the interplay with other fields been like?" Etc.*

The authors emphasize that experiences with concrete historical examples are essential to the development of a solid understanding of how mathematics has developed culturally and socially.

Overview and judgment concerning the *nature of mathematics as a subject area* (OJ3) could be addressed through the following questions (p. 77):

- *"What is characteristic of mathematical problem formulation, thought and methods?"*
- *"What types of results are produced and what are they used for?"*
- *"What science philosophical status does its concepts and results have?"*
- *"How is mathematics constructed?"*
- *"What is its connection to other disciplines?"*
- *"In what ways does it distinguish itself scientifically from other disciplines?" Etc.*

As these examples of questions show, having mathematical overview and judgment is not only a matter of possessing a certain knowledge. Some of the questions do not have answers that can be confirmed objectively based on their truth-value, but will be a matter of a person's beliefs. For example, if you ask a person what is required to be able to use mathematics, or how mathematics is constructed, the answer will depend on his or her experiences with mathematics, view of the nature of mathematics, knowledge about what mathematics can be used for, perspective of the relevance of mathematics, etc. Moreover, the person asked will most likely be quite aware that other people may answer differently. In contrast, the answer to a question concerning the actors involved in the historical development of mathematics will most likely include knowledge, for example, about famous mathematicians and mathematical concepts.

From a teaching perspective, a focus on these three forms of mathematical overview and judgment can thus contribute to a development of students' knowledge and insight of the nature of mathematics and its role in the world as well as their beliefs about mathematics as a discipline. This work is of course a complex matter, but considering the importance of building students' beliefs on evidence (Green, 1971), it must necessarily involve concrete examples of at least the application and the historical development of mathematics, and perhaps also of the nature of mathematics as a subject area.

Intermezzo

To avoid conceptual confusion, I briefly turn to a clarification of related terms. In this dissertation, the terms "mathematics as a discipline", "mathematics as a school subject", and "nature of mathematics as a subject area", are all used to describe connected, but different aspects of mathematics.

As elaborated in section 2.2.2, *mathematics as a discipline* describes mathematics in a non-school setting, and includes issues related to the societal and cultural role of mathematics in the world as well as a science.

Mathematics as a school subject or *school mathematics* is used to describe the character of the subject matter taught in school, with all that it entails. A person's beliefs about mathematics as a school subject thus include beliefs about how mathematics is learned and taught, what a mathematical task is, what mathematics in school is all about, what the socio-mathematical norms are in a classroom, what makes a student successful in mathematics, etc.

Nature of mathematics as a subject area defines, as mentioned above, the third form of overview and judgment and covers the aspects and characteristics connected to the *scientific* discipline of mathematics (i.e., mathematical problems and methods, philosophical and epistemological issues, etc.). Thereby, it constitutes part of mathematics *as a discipline*. It is, however, *not* synonymous with mathematics as a *school* subject.

2.3.2. Overview and judgment as a representation of mathematics as a discipline

By qualifying mathematical actions and contributing to "a more balanced and multifaceted image of mathematics as a discipline" (Jankvist, 2015a, p. 42), the purposes of developing students' overview and judgment are thus of an overall nature related to students' *Allgemeinbildung*, critical sense, and democratic competence. Furthermore, the purpose is to insert mathematics into the contexts in which it occurs, whether it is societal, historical, or intra-mathematical. The part of overview and judgment that consists of beliefs are thus similar to the dimension of the belief system concerning mathematics as a discipline, as it (in short) includes aspects of mathematics that are not as such related to the school subject. When aiming to develop students' beliefs about mathematics as a discipline, the three forms of overview and judgment thus provide a framework for including this work in the teaching. Along with an explicit focus on developing students' beliefs, this framework also includes a development of students' knowledge about mathematics as a discipline that may function as evidence on which their beliefs can be built. The questions used by Jankvist (2015a) to define students' beliefs about mathematics as a discipline (see section 2.2.3) are to some degree inspired by the exemplifying questions above. Furthermore, the three forms of overview and judgment can all be found in Jankvist's overall definition of mathematics as a discipline: "mathematics as a pure science [OJ3], an applied science [OJ1], a system of tools for societal practices [OJ1 + OJ2] as well as the philosophical [OJ3] and epistemological nature of mathematical concepts, theories [OJ2 + OJ3], etc." (Jankvist, 2015a, p. 45).

2.3.3. Overview and judgment in the Danish curriculum

The paragraph in the Danish mathematics curriculum regarding the overall purpose of mathematics education in primary and lower secondary grades (1-9) can, to a high degree, be linked to the notion of mathematical overview and judgment. The paragraph consists of three sections, presented below (Danish Ministry of Children and Education, 2019a, p. 3, my translation). After each of them, I elaborate on possible connections to:

> <u>Section 1:</u> *In the subject of mathematics, students must develop mathematical competencies and acquire skills and knowledge so that they can act appropriately in mathematics-related situations in their current and future daily life, leisure, education, work, and social lives.*

The phrase "act appropriately" is essential to this purpose. As Niss (2020) points out, it can be interpreted as being able to "with overview and judgment, independently and with impact, choose and complete various mathematical approaches to an appropriate treatment of a situation", and also to "reflect on what is going on as well as their mathematical actions". There is a clear signal in the last phrase concerning the context of the situations where students should be able to act appropriately, as it is highly connected to non-school settings or life outside school. Thus, mathematics should be taught with the clear purpose of enabling students to apply it to situations that exist outside the world of mathematics and mathematics education—which is a central aspect of OJ1.

> <u>Section 2:</u> *Students' learning must be approached so that they independently, and through dialogue and collaboration with others, experience that mathematics requires and promotes creative activity, and that mathematics contains tools for problem solving, argumentation and communication.*

This section indicates that mathematical methods and tools can be transferred to extra-mathematical situations and that the students should be familiar with these methods and tools in order to apply them and benefit from them. The phrase "creative activity" and the explicit mention of problem solving and argumentation indicate a dynamic view of mathematics that lies in the relativistic end of the spectrum (cf. section 2.2.3). Hence, OJ3 is strongly represented here. Furthermore, the section clearly encourages that learning is built from *experience*, thus supporting the thoughts behind the design of the intervention in this study.

> <u>Section 3:</u> *The subject of mathematics must contribute to the students experiencing and recognizing the role of mathematics in a historical, cultural, and societal context, and that the students can relate*

evaluatively to the application of mathematics in order to take responsibility and exert influence in a democratic community.

This last section of the paragraph almost includes direct references to overview and judgment. First, "a historical, cultural, and societal context" can easily be connected to OJ2 (historical and cultural), and OJ1 (cultural and societal). Second, "the application of mathematics" is mentioned explicitly. The overall notion of overview and judgment is strongly related to democratic competence in terms of being able to critically evaluate and reflect, as well as being an active citizen in a complex society, and can thus be connected to the purpose formulated in the last part of the section.

As stated in chapter 1, there is thus a strong resonance of the notion of overview and judgment in the stated purpose of mathematics education in primary and lower secondary grades, and the essence of it is also well-represented in the guiding curriculum (which as mentioned in section 1.2.1 can be adjusted locally). However, when studying the achievement goals for the individual grades, the words 'overview' and 'judgment' only occur sparsely. For example, the historical development of mathematics is only mentioned in a suggestion to include the history of numbers in the teaching. In contrast, the mathematical competencies[5] are very well represented with specific goals for each of them after grade 3, 6 and 9, similar to the mathematical content areas. The words 'overview' and 'judgment' are used as part of the overall goals for mathematical competencies after grade 6 and 9 (figure 10):

Competency goals

Field of competence	After Grade 3	After Grade 6	After Grade 9
Mathematical competencies	The student can act appropriately in situations with mathematics	The student can act with overview in compound situations with mathematics	The student can act with judgment in complex situations with mathematics

Figure 10: Competency goals in the Danish mathematics curriculum ('Common Objectives') for grades 1-9. (Danish Ministry of Children and Education, 2019a, p. 8).

As part of the Danish competencies framework, the notion of mathematical overview and judgment is included in the Danish curriculum, but although the guiding curriculum is elaborative in regard to the aspects included in this notion, the binding objectives thus only include them on an overall level. This is perhaps one of the reasons that it appears to be an overlooked part of the mathematical meta-competence by both teachers and students. Regardless of the reason, the intentions regarding the

[5] The eight mathematical competencies were in 2015 reduced to six with the combination of the reasoning competency and the thinking competency, as well as the representation competency and the symbol and formalism competency.

development of students' overview and judgment, which lie in the purpose of the subject as well as in the guiding curriculum, do not seem to be fully realized in practice.

Choosing the notion of overview and judgment as a framework for this study is thus primarily based on three arguments:

- The aspects of mathematics that are included in the belief dimension concerning mathematics as a discipline can be represented by the three forms of overview and judgment (cf. section 2.3.2).
- The notion of overview and judgment is already present in the curriculum as part of the competencies framework.
- There is a need for articulating and exposing the importance of developing students' overview and judgment, as well as investigating possible teaching approaches to support such a development.

Although the curriculum recommends introducing and implementing mathematical overview and judgment in the mathematics teaching throughout primary and secondary school, the notion as a whole (overview *and* judgment) is only included in the competency goal from lower secondary school classes (cf. figure 10). However, when theoretical considerations on belief development is taken into account, there might be valid reasons and arguments for initiating this work earlier, perhaps even in the first years of school. As described in section 2.1.3, early developed beliefs become filters through which new information is perceived (Pajares, 1992). An early focus on the students' beliefs about mathematics, in all its aspects, will thereby contribute to the students' perception of and approach to the subject in their entire schooling, education and life. In contrast, if this focus is not included in the teaching, the students will still develop beliefs. These will, perhaps, not be based on evidence and they might not support the students' learning, interest, motivation, or feeling of relevance. Moreover, such beliefs might not match a competency-oriented teaching approach. If the development of the students' mathematical overview and judgment is not prioritized, one of the two pillars that constitute the students' mathematical meta-competence will be missing from their education.

2.4. Changing beliefs and the role of reflection

Rösken et al. (2011) point out that most research studies on students' beliefs are descriptive, reporting for example typical beliefs. Fewer studies address the process of changing or developing students' beliefs. According to Grootenboer and Marshman (2016), this process is not well understood. One of the reasons behind this imbalance might be that beliefs are considered rather difficult to change and doing so requires both time and effort (e.g., Pajares, 1992; Schoenfeld, 2015).

To approach an understanding of how beliefs can be changed, it might be fruitful to recall how they are formed. As described in section 2.1.3, people infer and develop their beliefs from experience, and these early formed beliefs then function as filters through which subsequent information and experiences are perceived. When a person then meets new information about issues on which they have already established beliefs, a two-way evaluation takes place: existing beliefs are compared to and assessed against new information, and new information is conversely compared and evaluated against the already established beliefs.

By networking belief theory with theories about knowledge (Appleton, 1997), Rolka and Roesken-Winter (2015) describe three potential outcomes, when presented with new information:

1. **Identical fit**. It is possible to understand the new information based on existing beliefs, and the beliefs remain unchanged.
2. **Approximate fit**. Overall, the new information is related to the existing beliefs, but there are details that are unclear or in conflict. Still, it is possible to assimilate the new information without giving up on existing beliefs. Perhaps new beliefs are adapted into the belief system.
3. **Incomplete fit**. The new information cannot be explained on the basis of existing beliefs, and a cognitive conflict occurs. Generally, this can lead to several outcomes:
   - A rejection of the new information.
   - A subjective understanding or modification of the new information, so that it may fit the existing beliefs.
   - A modification of existing beliefs, so that it is possible to include the new information.
   - A rejection of existing beliefs, replacing them with new beliefs based on the new information.

The third outcome—the incomplete fit—thereby entails the potential for changing beliefs and is thus of interest in the present context. Of course, the changing process is complex and involves both psychological and sociological aspects. For example, repeated evidence of new information, which was initially rejected, may eventually instigate a change in beliefs. Yet, the latter option—a *change* in beliefs—is always the last alternative (Pajares, 1992). According to Nisbett and Ross (1980), individuals try to avoid a change in beliefs for as long as possible, for example, by attempting to make the conflicting information fit their existing beliefs by ignoring or twisting it. This results in the previously mentioned *perseverance phenomena* (cf. section 2.1.3). However, as mentioned in section 2.1.4, some (typically central) beliefs are more stable and resistant to change than others (more peripheral). The question of beliefs' stability has been reviewed by Liljedahl et al. (2012), who find that although beliefs generally are considered stable (even with ambiguous definitions of the construct of 'stability'), it is not impossible to change them under the right conditions or—in the context of teaching—through the right intervention. As mentioned in section 2.1.4, the stability of beliefs is connected to their level of dependency, their degree of conviction, as well as the way in which they are held (Green, 1971). Beliefs never exist independent of a belief system, so when a belief is changed, it affects the beliefs to which it is related. Primary beliefs often form the foundation for several derived beliefs and thus have extensive relations. The longer a person possesses a belief, the more robust it becomes, often even if contradicted by new evidence. There might also be strong psychological mechanisms related to (central) beliefs. They may be attached to memories, function as a basis for other strong beliefs, or serve self-protective, ego-enhancing or social control purposes (Op't Eynde et al., 2002).

A change in beliefs requires that the individual discard presumptions that (s)he holds to be true. Certain conditions are therefore required for such a change to take place. One of them is an emotionally safe context, as pointed out by Goldin et al. (2009). Within a classroom, it is thus essential to have a supportive and accommodating learning environment, making the students feel comfortable when they make mistakes, express doubts and insecurities and attempt to obtain coherence in their belief systems.

Liljedahl (2011) argues that the theory of *conceptual change* (emerging from the ideas of Kuhn, 1970), where a current conception is first rejected and then replaced by another, may be applied as a theory for *changing conceptions* (which is used as a synonym for beliefs). The theory of conceptual change concerns situations where learning is a result of experiences that contradict one's original conception about something (the incomplete fit), and not learning through integration of ideas (identical of approximate fit). It is thus related to cognition, but studies show that it may also be applied to affect, for example beliefs (Liljedahl et al., 2007). A contradictive experience is in the theory of conceptual change called a *cognitive conflict*, parallel to the 'incomplete fit' described above. Tall and Vinner (1981) describe a cognitive conflict in relation to students' mathematical concept development as the situation that occurs when conflicting concept images are evoked simultaneously, creating a sense of confusion. A corresponding phenomenon is described by Piaget (1964), who uses the term *disequilibrium* about situations where a new and unexpected experience causes confusion and frustration, leading to a change in cognitive structures in order to regain *equilibrium.* Transferred to theory on belief systems, the state of equilibrium can be seen as equivalent to the point made by Op't Eynde et al. (2002), stating that people strive for coherence in their belief system.

Referring to research in science education (e.g., Posner et al., 1982), Rolka and Roesken-Winter (2015) identify three conditions for *conceptual change* in relation to beliefs:

1. Students possess 'misconceptions' that are formed through lived experiences without formal instruction.
2. Students must be dissatisfied with their existing conceptions.

3. A cognitive conflict initiates a rejection of the current beliefs, which is necessary before new beliefs can be adopted.

Where the first of these conditions refers to evidentially held beliefs as a prerequisite for change, the second addresses the students' role in the process. For a change to happen, it is necessary that the students are aware of the incomplete fit between their existing beliefs and the experiences that they are presented with. Thereby, their existing beliefs need to be explicit (or conscious). Many of our beliefs are held at an unconscious (or implicit) level and have never been articulated (Furinghetti, 1996). If we are not asked about our beliefs or presented with information that is related to them, we do not necessarily think about them or become aware of them (Lester, 2002). Also Schoenfeld (2015) points out that when beliefs are not recognized, the difficulty of changing them increases. A way for unconscious beliefs to become conscious can be to be given the opportunity to articulate them, thus 'awakening' the beliefs by external motivation (Furinghetti, 1996). The third condition regards the process leading to a replacement of beliefs. In a teaching context, this process thereby includes considerations about which activities or approaches might induce cognitive conflicts, as well as which evidence might support the development of new (perhaps specific) beliefs.

In addition, it might be fruitful to consider which kind of beliefs are *important* to change compared to which are *possible* to change. There is a discrepancy in the fact that primary and central beliefs are the most influential and thus the most important and the most difficult to change. For example, Eichler and Erens (2015) find in their study of 51 secondary teachers that ,while their peripheral beliefs seemed to be modified during a traineeship, their central beliefs remained unchanged. As mentioned, peripheral beliefs are more accessible to change, but they do not necessarily influence the belief system on a level where it makes a difference. So how can an aim for changing students' beliefs about mathematics be approached in a way that on the one hand increases the chances of actual change, and on the other hand ensures that the change matters in the students' learning? Most studies on changing beliefs rely on approaches that incorporate *reflection* as a central element.

In the previously mentioned study by Jankvist (2015a), changing the beliefs of upper secondary school students was approached from three perspectives: (1) giving the students an opportunity to become aware of their beliefs about mathematics as a discipline by letting them answer a questionnaire, participate in individual interviews and focus group interviews; (2) presenting new evidence through two teaching modules dealing with exemplary cases of mathematics as discipline; (3) providing opportunities for reflection, e.g., through an essay task. Using the analytical assessment tool described in section 2.1.5, the results of this study showed a growth in the *consistency* between the students' related beliefs (e.g., stating that mathematics is invented and mentioning research as one of the sources for development in mathematics), a progress in their *justification* of their beliefs (e.g., using their knowledge about actual mathematical inventions to justify the statement of mathematics being invented), and an increased amount of *exemplification* (e.g., providing several examples of mathematical inventions), some of which could be ascribed to the teaching modules. Thereby, the intervention seemed to have contributed to the development of the students' beliefs about mathematics as a discipline, making them more conscious and based on evidence and reflection. By providing opportunities for articulating existing beliefs, experiencing new evidence and reflecting on the relation between these, the students thus seemed to have obtained a more nuanced image of mathematics as a discipline.

In a study of pre-service mathematics teachers' beliefs, Cooney et al. (1998) identified four different types of responses toward changes in beliefs, which might also apply to students, because of their general nature:

- The *isolationist*, who possesses beliefs that are separated and clustered, rejects any beliefs in conflict with his or her own, and is not capable of accommodation. The reaction of the isolationist resembles the reaction of 'non-engagement'.
- The *naïve idealist*, who uncritically accepts the ideas and beliefs of others in the search for agreement and harmony.
- The *naïve connectionist*, who has a more reflected view of the world and is able to accommodate and reformulate their beliefs but does not resolve conflict or differences. Both the naïve idealist and the naïve connectionist might react accordingly by 'building a new set of ideas.
- The *reflective connectionist*, who does in fact, in contrast to the naïve connectionist, resolve conflicts in their beliefs through reflective thinking and thus reflect the third reaction above: 'reforming existing beliefs'.

Mason and Scrivani (2004) investigated how fifth grade students' beliefs about mathematics can be developed by changing the classroom learning environment. The 12-session intervention focused on developing and refining the beliefs that might be of a naïve or maladaptive nature through three initiatives: (a) a re-negotiation of the socio-mathematical norms in the classroom, emphasizing the students' own responsibility for their learning, the teachers' role as facilitator and the properties of mathematical problems and problem solving; (b) an establishment of the socio-cognitive interaction between the students through small-group assignments and discussions, giving room for articulations and reflections on problem solving; (c) an introduction of non-routine problem situations. The measurement of the students' beliefs included a beliefs-questionnaire and a self-report concerning their change in beliefs. The results showed a positive development in the students' beliefs that seemed to become more advanced and articulate.

These examples indicate that essential elements in the process of changing beliefs are:

- *articulation* of beliefs
- *experiences* that contradict existing beliefs
- *reflection* on the relations between existing beliefs and new experiences

Thereby, it is not only the *beliefs* that are addressed, but also the *relations* between both new and existing beliefs in a belief system. By making the relations between beliefs an object of interest and reflection, a change in peripheral beliefs might lead to a change in central beliefs. As stated by Wilson and Cooney (2002) in relation to teachers' beliefs:

> *[…] attention to beliefs about teaching in general may result in certain peripheral beliefs (Green, 1971) but lack connections to more centrally held beliefs. It is through the act of reflecting on specific events that those centrally held beliefs can be affected in fundamental ways (Wilson & Cooney, 2002, p. 142).*

Although this quote concerns teachers' beliefs, the consideration about the importance of reflection in the process of connecting experiences to centrally held beliefs, is not restricted to teachers and may thus apply to students as well. Still, there is no guarantee that the changes in beliefs will last, and even in longitudinal studies, it is difficult to investigate the durability of new beliefs. According to Schoenfeld (2015), the chances for beliefs lasting increase when they are based on experience, and Jankvist (2015a) underlines that "the higher the level of reflection associated with changes in beliefs, the larger the probability that the changes may last" (p. 44).

Dewey (1933) defines reflection—or reflective thinking—as an "active, persistent and careful consideration of any belief or supposed form of knowledge in the light of the grounds that support it, and further conclusions to which it leads" (p. 118). Cooney et al. (1998) point out that reflection directs a person's attention towards the relevance of context, making his or her beliefs more nuanced and open to different perspectives. For Rokeach (1968), the attention to context indicates the open-mindedness of a person, as a closed-minded person sees the world in a perspective, where context is irrelevant. Because evidentially held beliefs are based on experience and reason, they are often related to certain contexts, thus making this form of beliefs more nuanced. The act of reflection reinforces the relation between beliefs and context and contributes to the open-mindedness of a person.

Reflection, thereby, appears to be a crucial element of belief change in a general sense (e.g., Cooney et al., 1998; Erens & Eichler, 2013; Ernest, 1989; Philipp, 2007) for several reasons:

- Reflection contributes to an awareness of existing beliefs.
- Reflection establishes connections between existing beliefs and new experiences.

- Reflection helps resolve potential cognitive conflicts.
- Reflection strengthens the relations between the beliefs in the belief system, thereby increasing the chances of lasting beliefs.
- Reflection draws the attention to the relevance of context, making beliefs nuanced and encouraging critical thinking.

It thus becomes essential to include the element of reflection in the design of an intervention aiming to develop, modify or change students' beliefs.

2.5. The "ideal" beliefs – can they be defined?

In continuation of the points found in the literature review in section 2.2.3 concerning the quality of beliefs and a possible ranking of which beliefs might be more fruitful than others, I find it central to discuss whether or not ideal beliefs can and should be defined—and if so, then to what extent, and how.

Lampert (1990) identifies the dualistic beliefs about school mathematics in general:

> *Commonly, mathematics is associated with certainty: knowing it, with being able to get the right answer, quickly (Ball, 1988; Schoenfeld, 1985a; Stodolsky, 1985). These cultural assumptions are shaped by school experience, in which doing mathematics means following the rules laid down by the teacher; knowing mathematics means remembering and applying the correct rule when the teacher asks a question; and mathematical truth is determined when the answer is ratified by the teacher. Beliefs about how to do mathematics and what it means to know it in school are acquired through years of watching, listening, and practicing. (Lampert, 1990, p. 32)*

On the basis of this comment, Schoenfeld (1992) presents a list of typical student beliefs about the nature of mathematics (Figure 11). Although Schoenfeld focuses on students' mathematical problem solving, these beliefs are all considered counterproductive for the students' mathematical understanding and an adaptive learning behavior in general, and they are certainly not compatible with the didactical philosophy behind the Danish competencies framework. Nevertheless, as Lampert points out, they are formed in the context of the classroom. Schoenfeld (1985) mentions "the presentation of construction as step-by-step procedures to be memorized", and a "strong emphasis on writing up mathematical arguments in a rigidly prescribed form" (p. 374) as examples of teaching actions that promote this kind of beliefs. Hence, there are certain beliefs that might not be desired in terms of the students' learning outcome and their approach to mathematics. As described in section 2.2.3, beliefs that belong in the relativistic end of the spectrum seem to be related not only to students' performance level, but also their interest and motivation.

Typical Student Beliefs about the Nature of Mathematics
Mathematics problems have one and only one right answer.
There is only one correct way to solve any mathematics problem—usually the rule the teacher has most recently demonstrated to the class.
Ordinary students cannot expect to understand mathematics; they expect simply to memorize it and apply what they have learned mechanically and without understanding.
Mathematics is a solitary activity, done by individuals in isolation.
Students who have understood the mathematics they have studied will be able to solve any assigned problem in five minutes or less.
The mathematics learned in school has little or nothing to do with the real world.
Formal proof is irrelevant to processes of discovery or invention.

Figure 11. Typical student beliefs about the nature of mathematics (Schoenfeld, 1992, p. 359)

There is, however, a risk of bias or subjective judgment in determining which beliefs are better than others. As everyone else, researchers, policymakers and other stakeholders certainly also possess beliefs about the nature of mathematics and about which beliefs should be promoted in mathematics education—and as is the case for everyone, some of these beliefs might be held unconsciously or non-evidentially. As such, it might be unrealistic to assume that it is possible to identify an objective list of favorable beliefs. Furthermore, Rolka and Roesken-Winter (2015) emphasize that different perspectives on mathematics—e.g.. the classifications by Ernest (1989) or Dionne (1984)—can all be valuable when attempting to characterize *mathematics*.

In light of the points made in this chapter, it thus might be fruitful to define the ideal beliefs as reflected, nuanced, and based on evidence. Thereby, the quality of students' beliefs is valued based on the way in which they are held. When students base their beliefs on experience and reflect on the relationships between their beliefs, they are, to a larger extent, able to evaluate new information and experiences with an open mind, and to potentially transfer their beliefs to other situations because they pay attention to the relevance of context. For the purpose of this study, it thereby becomes an aim in the design of the intervention to provide opportunities for the students to form or develop such beliefs, thus seeing the *reflective connectionist* as an ideal. Furthermore, it becomes a guideline for the subsequent analysis of the collected data and a tool for measuring the development of the students' beliefs.

With that said, teaching is inherently normative and aims for promoting a certain knowledge, certain skills and competencies, or certain beliefs. Designing an intervention aiming to increase the level of reflection in the students' beliefs inevitably entails considerations regarding the content, purpose and goals of the teaching and the individual learning activities. As stated in chapter 1, one of the aims of this study is to contribute to the students' *Allgemeinbildung*, democratic citizenship, and perceptions of the relevance of mathematics by developing and expanding their beliefs about mathematics as a discipline. This aim entails a relativistic perspective on mathematics, which is also found in the overall purpose of the subject of mathematics in the "Common Objectives". In this way, the notion of overview and judgment entails a relativistic perspective on mathematics.

Chapter 3: Methodology

Creswell (2009) presents a framework involving the intersection of three components for research design (figure 12):

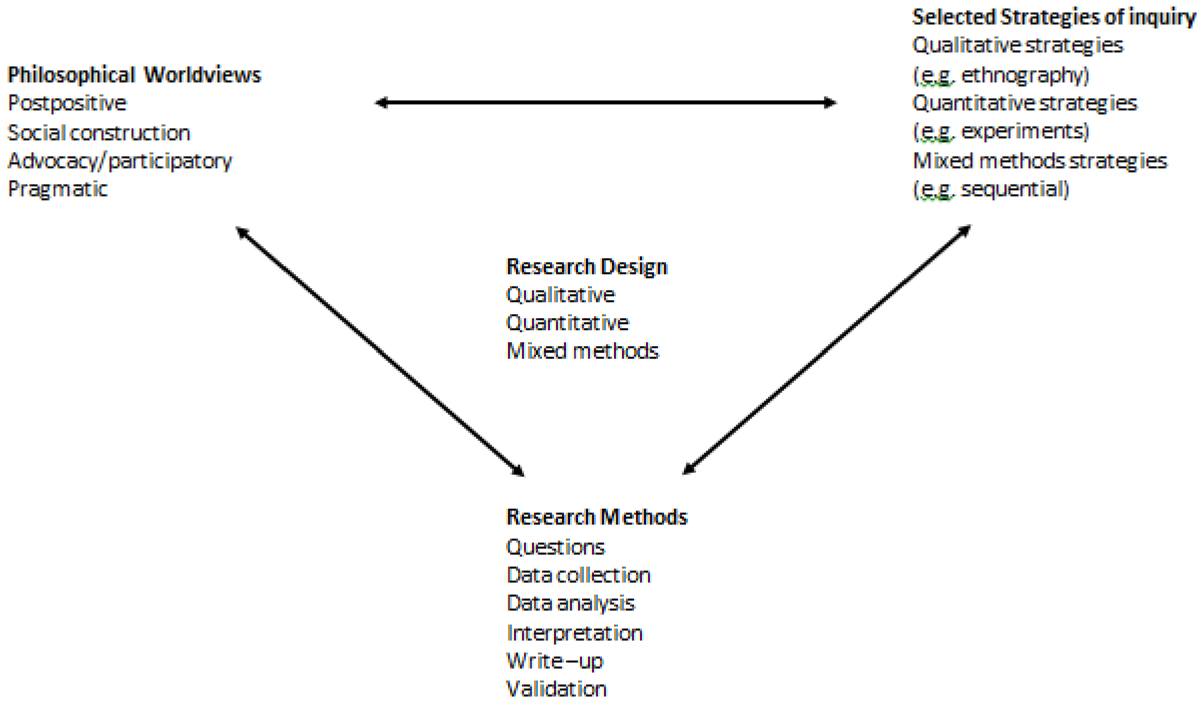


Figure 12: A framework for design—the interconnection of worldviews, strategies of inquiry, and research methods. (Creswell, 2009, p. 5).

In this chapter, I describe the methodological considerations of the study connected to two of these components: philosophical worldviews and strategies of inquiry. The third component, research methods, will be elaborated in chapter 4.

3.1. Philosophical worldview: constructivism

Designing a research study primarily consists of making choices and decisions. These decisions may be connected to conceptual framework, quality criteria, practical issues, ethical considerations, research questions, etc. Behind these decisions lies the researcher's perception of not only what constitutes good research, but also what is important in the world and in the field of interest. In other words: the researcher's beliefs. Creswell (2009) defines the term *worldview* as "a basic set of beliefs that guide action" (p. 6). Guba and Lincoln (1994) use the term *paradigm* to describe a similar concept, defining it as "basic belief systems based on ontological, epistemological, and methodological assumptions" (p. 107). In this study, I assume a *constructivist worldview*. The constructivist worldview builds on the assumption that individuals construct their subjective understanding of the world as they engage with it, which is in line with the theory connected to formation of beliefs, as described in section 2.1.3. Therefore, research based on a constructivist worldview is often related to qualitative research as it seeks to investigate the complexity of the participants' views, including the context in which they are situated (Creswell, 2009). The constructivist perspective—that the participants' understanding is constructed while engaging with the world—is significant to the epistemological view (Collin & Køppe, 2014). It means that understanding and meaning is created and developed even in the interaction between researcher and participant, "so that the 'findings' are *literally created* as the investigation proceeds" (Guba & Lincoln, 1994, p. 111, italic in original). In this study, I thus approach the phenomenon of interest—the students' beliefs—in a way that somehow contributes to their formation and development. Hence, when I ask the students questions about mathematics during an interview, I influence their thinking about mathematics. On one hand, a student's existing beliefs influences how (s)he comprehends and receives the question, and on the other hand, the question and the student's articulation of the answer might contribute to the formation or modification of his or her beliefs. My phrasing, my tone, and my body language may all affect the student's perception of the question and possible expectations for an answer. Asking a question about an aspect of mathematics in a questionnaire might initiate the student's awareness of this aspect. The presence of a researcher with a camera in the classroom most likely affects the atmosphere, makes some students self-conscious, and puts special attention onto the research object. Therefore, a student's answer to a question, statement in an interview, or behavior in the classroom cannot be said to be an objective reflection of their beliefs, as they would look appear, had I not been there. Influencing beliefs is thus an unavoidable element of investigating beliefs. It is not possible to access students' beliefs without also affecting them, either in terms of influencing them, or in terms of evoking certain aspects of them—the latter being related to the before-mentioned *activation level* of beliefs (cf. section 2.1.5). This is an important point when characterizing the knowledge that is developed from this form of research.

Following my choice of conceptual framework, this premise in fact contributes to the process of developing students' beliefs, as my presence and questions initiate the students' awareness of aspects related to mathematics as a discipline, as well as an articulation of their existing and emerging beliefs. Still, it means that my answer to the first research question concerning the students' pre-beliefs must be understood with this condition in mind. Within the answers that the students give, already lies possible emerging beliefs or emerging alterations of beliefs.

A constructivist worldview, furthermore, recognizes that researchers have also formed subjective meanings of the world that influence their interpretation (Creswell, 2009; Guba & Lincoln, 1994). Combined with the fact that beliefs are not easily accessible and must therefore be inferred from interpretations of statements and actions, a central question asks what kind of knowledge about the students' beliefs is developed in this study, as interpretations are inherently subjective. An example of this is found in the process of analyzing the students' statements. As I investigate the students' beliefs about mathematics as a discipline, it is relevant to distinguish statements concerning mathematics *as a discipline* and statements concerning mathematics *as a school subject*. The students primarily develop their beliefs about mathematics as a discipline in a school setting and by the means of the school subject. Hence, they can only refer to mathematics as a school subject. The distinction thus relies on my interpretation: when do the students refer to the mathematics that they learn in school, and when do they refer to mathematics in a broader sense, e.g., the application, the historical development, or the nature of mathematics? As pointed out by Jankvist (2015a), the students' beliefs about mathematics as a discipline are primarily built in the context of a classroom and are thereby based on their beliefs about mathematics education (cf. figure 4, section 2.1.5). For some students, and in some classrooms, the disciplinary dimension of mathematics is not prioritized, or it is ignored or forgotten, and the students perceive mathematics merely as a school subject, and not as a discipline that contributes to daily life and society, has contributed to the development of this society, has a history of its own, and constitutes a scientific discipline. In such cases, the students might develop beliefs about mathematics as a discipline that are identical to their beliefs about the school subject, which (possibly) makes it difficult for them to see the relevance of learning mathematics in a broader perspective. When I analyze the students' statements and behavior, it is thus my responsibility to interpret whether aspects of mathematics as a discipline are included in their beliefs. For example, a statement like "you only learn mathematics so that you can pass the exam" is a clear example of beliefs about mathematics as a discipline that is limited to school mathematics. In contrast, "mathematics is used for many purposes in the world, for example our tax system, engineering and computer programming" is clearly related to the application of mathematics outside school. However, a statement such as "perseverance is important in mathematical problem solving" can be interpreted as a reflection of socio-mathematical norms in the classroom or of insight into scientific mathematical methods. In those cases, I need to consider other perspectives or types of data in my interpretation. Related statements, the context of the statement, observations of the student in the classroom, or discussions with the teacher might provide a more nuanced understanding of the student's beliefs and thus contribute to a more informed and qualified interpretation.

In this way, triangulation of data types plays an important role in the interpretation and understanding of the students' beliefs, as it makes it possible to consider several perspectives and forms of expression. The risk of blind spots thereby decreases, and the different types of data will contribute to different aspects of the students' beliefs, while also making it possible to compare the data that cover the same aspects. The specific methods and data types used for triangulation in this study will be elaborated in chapter 4.

As already described, this study aims to understand the developmental process of students' beliefs, as well as to make the students more conscious of their beliefs, and to make their beliefs more nuanced, reflected, and evidentially held. Guba and Lincoln (1994) point out that within the constructivist paradigm,

> *[t]he aim of inquiry is understanding and reconstruction of the constructions that people (including the inquirer) initially hold […]. The criterion for progress is that over time, everyone formulates more informed and sophisticated constructions and becomes more aware of the content and meaning of competing constructions. (Guba & Lincoln, 1994, p. 113)*

It is not difficult to see the resemblance to the aims of this study. The methodological approach thus needs to include the opportunity for both understanding and development, not only concerning the students' beliefs, but also in relation to methodological considerations such as figuring out how beliefs can be developed and by what means. It is, furthermore, essential that the strategy for inquiry may accommodate that beliefs are formed and developed through experiences in the context in which they are situated (in this case: the classroom). As elaborated in the following section, these requirements can be met in a Design-based Research approach.

3.2. Strategy for inquiry: Design-based Research

> *If you want to change something, you have to understand it, and if you want to understand something, you have to change it. (Gravemeijer & Cobb, 2006, p. 45)*

The quote above illustrates how Design-based Research (DBR) emerged from a desire to, at the same time, develop theories about teaching and learning as well as change and develop practice. It is a family of methodological approaches in which design and research are interdependent (Cobb & Gravemeijer, 2008), making it possible to meet the complexity of teaching that cannot be included when studying learning, cognition or knowledge as isolated entities. These are inextricably linked to a context and must be studied in this context to provide a complete understanding. Where the first research question in this study seeks understanding, the second research question focuses on development of both the teaching and the students' beliefs, and also on an understanding of this development. The DBR-approach is thus applied in the second research question. However, the findings from the first research question inform the *design* on which the research is based.

As mentioned in section 1.7, the second research question concerning the development of students' beliefs about mathematics as a discipline is addressed through a *longitudinal study* using a *Design-Based Research* (DBR) approach (Bakker & van Eerde, 2015; Barab & Squire, 2004; Cobb & Gravemeijer, 2008; diSessa & Cobb, 2004). Several considerations have been taken in relation to these methodological choices. The fact that changing beliefs can be very time-consuming (c.f., section 2.4, Green, 1971; Pajares, 1992) is the main reason for letting the study proceed for as long as possible within the boundaries of a three-year Ph.D. scholarship. Beginning in August 2019 and ending in June 2021, the intervention lasted for two school years. Furthermore, a change or a development in students' beliefs requires a consistent and long-term exposure to examples that can function as evidence for the desired beliefs. Thereby, this exposure should happen consistently in the mathematics teaching, thus involving the teachers of the two classes. In this study, the exposure happens through an intervention with an increased focus on the students' overview and judgment, in the form of principles for teaching that have been designed based on beliefs theory. A DBR approach makes it possible to collaborate with the teachers as partners (Barab & Squire, 2004), while not only implementing a fixed design but allowing the design to be improved along the way (cf. section 3.2.1). In addition, the dual nature of DBR joins theoretical and practical aspects of a research study (Cobb et al., 2017). In this way, the different kinds of expertise as well as the different aims and contributions of the researcher (theory) and the teachers (practice) are combined and utilized.

An overview of the intervention and the data collection is illustrated in figure 13.

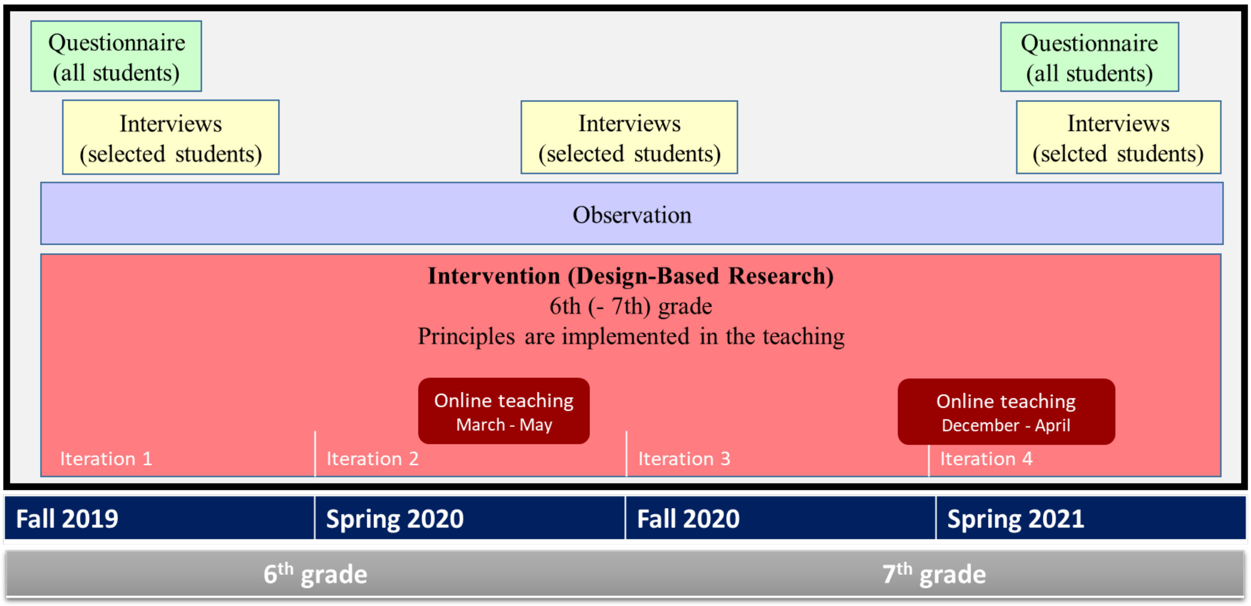


Figure 13: Overview of the intervention. Elements and timeline.

3.2.1. Features in DBR

Other related approaches are design research, design studies, and developmental research. I use the term *Design-based Research* to emphasize the element of *research* (Bakker, 2018). The design, on which the research is based, is not the key focus of the study, but functions primarily as a means to investigate the character and the development of students' beliefs about mathematics as a discipline, and secondarily as a goal in itself.

Certain features characterize DBR (Bakker & van Eerde, 2015; Plomp, 2013). Separately, they are not unique for this approach, but taken as a whole, they constitute a defining frame. Below, I list five features of DBR that are especially important for this study, along with general comments as well as comments on how they each are met:

Dual purpose: Firstly, DBR aims to develop, test, and adjust presumptions about learning processes, thereby generating understanding and theoretical knowledge. The theoretical perspective focuses on generalizability through the applications and development of design theories, which includes domain-specific theories, design frameworks and design methodologies. These theories explain why a design works and suggest how it may relate to other contexts. Secondly, it has a pragmatic purpose; aiming to study and develop a specific design (Cobb et al., 2017). The focus is to improve practice through engaged, solution-oriented development and research. The pragmatic criteria for success are applicability of the design and knowledge about practice. Figure 14 shows how theoretical perspectives in combination with pragmatic requirements from different stakeholders (government, school, teachers, parents, students, etc.) form the basis of the design.

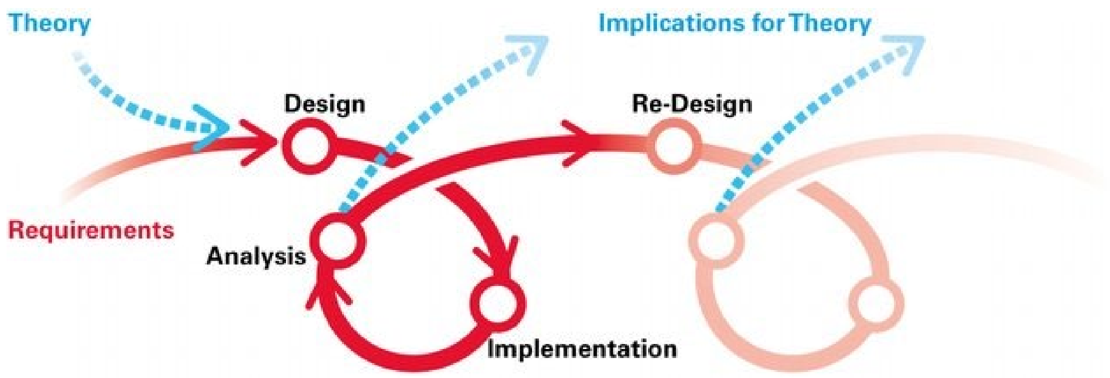


Figure 14: Design based research as an ongoing process of innovation. (Fraefel, 2014, p. 9)

In this study, the theoretical perspective, which includes theoretical constructs related to OJ and beliefs theory, plays an essential role both in the design of the intervention, in the teaching principles, and the retrospective analysis of the intervention's iterations. Furthermore, the *theoretical* aim of the project is to generate understanding of, or knowledge about, the character and development of students' beliefs and their overview and judgment. In relation hereto, methodological approaches to generate this knowledge are explored, i.e., how beliefs are investigated both in relation to their character and their development. The *pragmatic* perspective is included in the design of the teaching principles, primarily through the involvement of the teachers as collaborators. During the intervention, the pragmatic perspective is largely present in the lesson planning and the implementation of the principles, and also in the evaluation and adjustment of the principles. The pragmatic aim of the project is to develop principles for teaching that focus on developing the students' overview and judgment as part of a competence-oriented teaching.

Iterative nature: Due to the theoretical aim to understand the "ecology of learning" (Cobb et al., 2003, p. 12) as well as the pragmatic aim to change, DBR tests and revises the conjectures on which the design is based. Thereby, the conjectures can be evaluated and analyzed in order to understand, while practice can be changed through a revision of the design. The iterative nature of DBR can briefly be described as cycles of designing, implementing (testing the design), and analyzing, as illustrated in figure 14. Apart from the cycles of phases, figure 14 also shows how the retrospective analysis related to each iteration not only provides information to the adjustment of the design, but also generates theoretical knowledge (Bakker & van Eerde, 2015; Cobb & Gravemeijer, 2008).

Phase 1: Preparation and design

In the initial phase, the intention of the study is clarified. A wish to meet certain challenges or solve a certain problem is often the root of this step, but both practitioners and researchers can be the instigator. From this intention, an intervention is designed based on the existing knowledge of the participants in the form of theoretical knowledge, existing research, and practical experience. Furthermore, a clarification of requirements, as well as logistical and physical possibilities is necessary to frame the design. Also, knowledge concerning students' prior knowledge and learning potential, present practice, etc. must be considered. The design phase also involves thought experiments and hypotheses regarding both the intention, the implementation, and the outcome of the intervention. It involves setting up clear learning goals and designing activities, but it also prompts the participants to thoroughly consider the connection between their intention and their design. This work is often called the *hypothetical learning trajectory* (e.g., Bakker & van Eerde, 2015), which includes learning goals, learning activities and a hypothetical learning process. As mentioned in the section concerning features of DBR, the hypothetical learning trajectory (HLT) can change during the intervention.

Phase 2: Test / experiment / implementation

When the design is ready, it is tested in the second phase. The learning activities are implemented in the teaching with the HLT as a guideline. It is also a guideline for choices taken in relation to the focus of data collection. An important element is debriefing among the participants, which enables an ongoing and documented adjustment of the HLT.

Phase 3: Retrospective analysis

In the last phase of a cycle, the design is evaluated and data is analyzed. The goal is to close the gap between the intended, the implemented, and the realized design. Hence, the participants consider to what extent the reality matched their intention and their expectations and how the design can be adjusted. The purpose of this phase is thus to improve the design rather than to show that it works. Furthermore, the results of the analysis in this phase contributes to theory which, again, is considered in the adjustment of the design, thereby letting theoretical knowledge grow out of practice while remaining part of practice. The retrospective analysis leads to a redesign of the intervention, which is then implemented in the next iteration.

In this study, principles for teaching are designed in collaboration with the teachers, based on the three forms of OJ as well as on theory of beliefs and their development. Based on collaborative discussions, these principles are then implemented in the teaching by the teachers. The intervention is structured in four interventions, each of them lasting one semester. In each iteration, the principles are tested and subsequently evaluated and adjusted or refined. Also, pragmatic issues are taken into consideration in these adjustments, such as formulation of the principles, time perimeters, or structure of the lessons. This all contributes to a qualification of the design, which is then implemented and tested in the following iteration.

Interventionist nature: As stated above, the dual purpose of changing as well as understanding makes it necessary to study phenomena connected to learning in a context (e.g., the students' beliefs), while changing this context at the same time. To study learning processes as isolated variables (e.g., in laboratories) will give an incomplete understanding and only observing learning in a context collides with the wish to change.

Given that this study is highly based on the longitudinal intervention, it certainly fits this feature. In line with the constructivist view, the students' beliefs about mathematics as discipline are studied in the context of which they are primarily developed. Simultaneously, these beliefs are sought to be developed through the increased focus on OJ in the implemented principles. Thereby, an understanding is built of the character of the students' beliefs and of the developmental process of changing them, while addressing a wish to develop their beliefs in an intended way. Also, a methodological understanding of how beliefs and their development can be investigated is formed.

Collaboration: In DBR, the participants are not subjects, but co-participants who join in shared reflections. This means that they are not just "data providers", but also actively contribute to the customization of the design. The researcher has an intention with the research, but the participants' identification of the problem is also important. There must be a real need for change, so that theory can be applied to a pragmatic situation and actually do "real work".

The teachers in this study are included both in designing and adjusting the principles and in the evaluation of the students' outcomes. The initial intention with the project came from me but, from the beginning of the collaboration, the teachers' have been involved in reflections about what kind of changes are desired in the students' beliefs and which problems these changes should address. Due to their practical expertise, the teachers (for the most part) planned the lessons based on the principles.

Prospective and reflective components not separated by experiment: Essentially, this feature indicates that even though each iteration is followed by an evaluation and a re-design, small adjustments can be made during the iterations. Since DBR, as mentioned, is a way to navigate in the complexity of the learning, there is continuously room for reflecting, updating, and adapting the design when it seems necessary. However, these changes must be well documented and argued for, so that they can be included and considered in a theoretical perspective in the actual evaluation at the end of the iteration.

In this study, this feature has been accommodated by regular debriefing sessions where the teachers and I have reflected on how the planned teaching related to what actually happened during the lessons, and how we might consider whatever challenges there might be in the future teaching. Furthermore, the teachers have made choices and changes "on the spot" during teaching. This was done sometimes to ensure that the principles were complied to and at other times to handle problems or challenges by deviating from the principles.

3.2.2. Criteria for selection of participants

Students' age and choice of school

The participating students were, at the beginning of the intervention, in 6th grade (11-12 years old). This age group was chosen based on the following considerations: The development of students' beliefs about mathematics is a continuous process that begins even before starting school through the influence of parents, other adults, media, and culture. If the belief dimension concerning mathematics as a discipline is not considered, there might be a risk that students primarily perceive mathematics as something that only belongs in school with no important part in the real world. For example, if the teaching in the first school years signals that mathematics is mostly about rules and formulas, that the criteria for success is to find the correct answer quickly, and that mathematics mostly is about working individually with training tasks, then this image is likely to form the basis for the students' central beliefs about mathematics. Since central beliefs can be quite difficult to change (Green, 1971), it is essential to address mathematics as a bigger picture than as a school subject from the beginning of school. Hence, there are on one hand good reasons for choosing students at a young age for this project. However, since the data collection is partly based on interviews, the students must on the other hand be able to express their thoughts and to reflect on their own beliefs. This requires a certain maturity. To accommodate these conflicting requirements combined with what was practically possible in regard to schools able to participate, the age group of middle school students was chosen. At the participating school, the students are transferred from another local school in 6th grade (maintaining the class composition), which thereby was the lowest possible age, when working with this school.

Teachers

A longitudinal study requires certain considerations from both researcher and participants. It is essential that everybody involved is committed to the project from beginning to end, and willing to prioritize it even in periods with demands from elsewhere. Before beginning the planning of the intervention, both the school principal and the participating teachers thus signed a contract, ensuring their continuous commitment during the two years (Appendix A). However, a contract does not guarantee a personal commitment and effort from the teachers. In my search for participating teachers, I was looking for a high level of professionalism, engagement in the subject of the project, and interest in professional development. Moreover, the participating school and the local area should be an example of an average Danish public school and within the average regarding socioeconomic status. I therefore contacted the municipal consultant in mathematics education with whom I have previously cooperated. She recommended several possible schools and teachers. I was familiar with two of these suggested teachers, since they had attended a one-year course in mathematical modeling that I taught the year before. In the course, they had both been engaged and keen on their own professional development. Hence, they already knew me, as well as my previous work related to beliefs about mathematics. Moreover, they were employed at a school that fit my criteria. When asked if they would consider participating in my two-year intervention study, they did not hesitate to accept, even before they were presented with the formal circumstances regarding estimated time, payment, etc. All in all, these two teachers seemed perfect for the project. Not long after the preliminary meetings with the teachers and the school principal, their colleague—a young teacher in her first year of service and still finishing teacher education—asked if she could be part of the project as well. This would mean that the intervention would take place in three classes instead of two, and hence provide not only more data, but also a larger diversity in both students and teaching practices. I therefore decided to include the third teacher. This decision turned out to be quite sensible. A few months into the intervention, one of the original two teachers announced her pregnancy, and went on maternity leave half way into the first school year. The substitute teacher did not wish to participate in the project, so I had to exclude this class from the rest of the intervention.

The two teachers who ended up participating were two very different practitioners. As mentioned, one of them (the teacher in the X-class) had no teaching experience; she was still attending courses to get her teaching diploma and she felt somewhat insecure about her teaching. Thus, she was both open and grateful for every idea, discussion and input presented in our collaboration. The teacher in the Y-class had 20 years of teaching experience; she also had an educational background in physics and a professional background as a teacher in the military. Moreover, she was the school's mathematics counselor. She thus had many valuable inputs to the project, yet she was still open to new ideas and very interested in her own professional development.

3.3. Quality criteria

Schoenfeld (2007a) does not apply the quantitative terms of validity and reliability but suggests that qualitative research is, instead, judged on three aspects, namely *trustworthiness*, *generality,* and *importance*. Seen as three different quality dimensions, they can be used to characterize research studies and their potential contributions. For example, a study might contribute with highly generalizable results without them being of particular importance, or, contrastingly, results of great importance that cannot be generalized. The following sections elaborate on these three dimensions.

3.3.1. Trustworthiness

The *trustworthiness* of a study indicates why its results are believable and how robust its conclusions are (Schoenfeld, 2007a). An aspect that contributes to a study's trustworthiness is its *descriptive power*, which is related to the applied theories and their capacity to focus on the phenomena being investigated. Also, the *explanatory power* plays a role, as it describes potential correlations and mechanisms between different phenomena. Rich, thick descriptions may contribute to both these aspects, as they explain how the study is conducted and why the data leads to certain conclusions. *Precision* and *specificity* is essential in such descriptions: "[T]he more carefully one describes both theoretical notions and empirical actions (including methods and data), the more likely one's readers will be able to understand and use them productively, in ways consistent with one's intentions" (Schoenfeld, 2007a, p. 87). Supported by Bakker and van Eerde (2015), Schoenfeld points out that *triangulation* of data is essential to the trustworthiness of qualitative research. Context plays an important role in qualitative research, and different circumstances may lead to different data. By looking at a phenomenon from different perspectives and through different lenses, information on the phenomenon may be challenged, confirmed or expanded.

The trustworthiness of this study hence lies in the quality and triangulation of the data as well as in the thick descriptions and transparency of the whole process. However, in order to ensure transparency, a researcher must be aware of all aspects connected to the choices made. Like every other person, a researcher has blind spots, preferences, agendas, and beliefs that are more or less conscious. For example, both my educational and professional background are strictly related to the field of education, as I have worked as a primary school teacher and as a teacher educator. This means that I am quite familiar with the context in which this study is conducted, the structural and curricular framework that is applied, and the practical circumstances and challenges that the participants face. It is thus easy for me to relate and adjust my research to the context. However, this familiarity may increase the risk of bias, as I might not be aware of implicit or taken-for-granted issues, habits, or organizational structures to which a researcher from another field might pay attention. I might be so familiar with the Danish teaching tradition that some alternative approaches do not occur to me as options. Moreover, my empathy with the teachers might even stand in the way of certain aspects of our collaboration, as I perhaps pay extra attention to their struggles and therefore more or less consciously avoid putting any pressure on them. For example, pressure in the form of challenging them in new teaching approaches that require extra preparation, expecting them to read relevant literature or questioning their teaching methods.

Another example is my academic predispositions. In the context of mathematics education research, I have—as mentioned in chapter 1—previously studied the phenomenon of mathematics anxiety and its connection to beliefs. Hence, I have a natural concern for students who exhibits signs of mathematics anxiety and an awareness of teaching methods and approaches that increases the risk of the students developing mathematics anxiety. This may be an advantage, for example, in the design of the teaching activities, but there is also a risk that I do not pay equal attention to other aspects. Throughout the entire study, I have thus strived to be aware of, and compensate for, the possibility of bias.

3.3.2. Generality

A study's *generality* (or generalizability, as Bakker and van Eerde (2015) call it) has to do with the range of the findings and to which extent they apply in other contexts. As mentioned, it can be difficult to determine the generality of qualitative studies and assess to which degree the results may apply in different settings, with different participants and a different culture. Hence, to enable others to judge what is specific for the particular study and what may be transferred to their context, it is necessary to make thick and thorough descriptions and documentation of the connection between context, design and findings (Lincoln & Guba, 1985). These descriptions may include factors such as setting, learning environment, prerequisites of participants, resources, and implementation strategy (Collins et al., 2004) as well as details of theoretical, methodological and analytical strategies and approaches. In the words of Creswell (2009): "Particularity rather than generalizability is the hallmark of qualitative research" (p. 190).

The generality in this study concerns whether or not a similar intervention would provide similar results in a different setting (e.g., in other middle school classes). Also, it must be considered if it is at all possible to conduct this intervention in another context. If so, which elements would be transferable and which would be restricted to apply in this specific context.

Seen from a constructivist view, the contextual element is particularly present in this study, as it concerns the students' beliefs, which as mentioned, are constructed and developed while engaging with the context. The question is then, what is specific about the context in this study, and what can be generalized. First, the participants can be said to always be specific. It is of course not possible to find middle school classes with the exact same composition of students with their personalities, prerequisites, performance levels or beliefs. Neither would other teachers implement the design exactly like the two teachers in this study. On the other hand, both the students and the teachers are not unusual or in any way 'odd' in a Danish context. Furthermore, the two teachers represent two quite different types of teachers, although they do work at the same school and in the same teaching culture, so to speak. This diversity might contribute to a higher degree of recognizability in terms of generalization. The same can be said about the selection of focus students who represent a variety of students, particularly regarding the level of reflection in their beliefs, and their attitude towards mathematics (cf. section 4.3). Still, it is unlikely that an implementation of this specific intervention design in another context would lead to the same results, as the context of a learning environment is so complex and includes so many variables that influence the implementation. For example, it is not possible to know what influence the periods of lockdown[6] have had on the development of the students' beliefs.

Second, the teaching can also be said to be specific for these two classes. The two teachers approach the intervention very differently, despite collaborative planning and common principles for teaching. Nevertheless, there are factors that these two classes share with each other and with other middle school classes in Denmark. Although mathematics teaching has local traits, it is also a *culture* that is shared nationally. Danish mathematics education shares regulatory provisions and curricula that set a framework for the teaching, which all teachers and schools must follow. It also shares a structure in the educational system that affects the prerequisites of both students and teachers. For example, the vast majority of Danish children enter school after four to five years in institutional day care, and teacher education takes place at university colleges with specific teacher training programs (not at a university). In addition, mathematics education in Denmark shares the historical development of how mathematics

[6] The COVID-19 pandemic caused two periods of lockdown, where the teaching was conducted online: March 2020 – May 2020 and December 2020 – April 2021, cf. figure 13.

has been perceived, applied, and taught. We have a shared understanding of what mathematics education traditionally has looked like and included, as we all have learned mathematics in school. Thereby, certain aspects in the mathematics teaching of the two classrooms in this study are expectedly cultural and thus generalizable. The overall character of the students' beliefs (cf. research question 1) may be related to this culture, as may some of the challenges connected to beliefs development. In this way, this study can be said to draw on *ethnographic tools* in its methodological approach (Green & Bloome, 1997), as part of the thick descriptions concerns both the local and the national culture of mathematics education. Thereby, others can assess which elements of the study apply to their specific context and which do not. For example, the methodological approaches in this study related to measuring students' beliefs may constitute a contribution that can be employed in other contexts that might not be restricted to middle school students.

3.3.3. Importance

The third dimension connected to the quality of a research study concerns its *importance* or its *relevance* to theory and practice. It is a question of whether or not its contribution matters and makes a difference, both locally and generally. The local contribution of this study is primarily pragmatic: it makes the students and the teachers aware of important aspects of mathematics education that have previously been overlooked. Moreover, it may contribute to an increased attentiveness to the students' beliefs, and the intervention, furthermore, contributed to an increased and appreciated collaboration between the teachers (supported by me as a researcher) leading to rewarding didactical discussions. In the range of possible generality, the study may also provide similar contributions in other contexts and raise awareness of the importance of developing students' mathematical overview and judgment in a competence-oriented mathematics education.

In terms of theoretical contributions, the findings of the study are relevant to belief research in at least three ways: (1) In relation to research question 1, the findings provide a characterization of middle school students' beliefs about mathematics as a discipline that has not previously been investigated in a Danish context. The study may thus be relevant in terms of beliefs as a reflection of teaching methods, in relation to an ambition of competence-oriented mathematics education, as well as to considerations concerning which beliefs might be desired, for instance in perspectives of education, society, democracy or life proficiency. (2) The findings of research question 2 contribute to further understanding of the development of students' beliefs, and the challenges connected hereto. The longitudinal design is not common in the research field concerning students' mathematics-related beliefs, and this study may add relevant perspectives to existing knowledge. (3) The methodological contributions of this study are potentially relevant to future investigations of students' beliefs and the attempt to modify, develop or change them. The longitudinal approach in combination with DBR and data triangulation may offer new perspectives on investigating and understanding the processes involved in developing students' beliefs.

3.3.4. Ethical considerations

The Danish Code of Conduct for Research Integrity (2014) is based on three basic principles: honesty, transparency, and accountability, which to a large extent follows the quality criteria listed above. Integrity in research builds on accuracy and openness in reporting objectives, methods, data, analysis, results, and conclusions. Furthermore, it requires thorough documentation of the research process and responsible data administration.

In relation to documentation, the process of the research study has been documented in detail in a project log, which includes all steps of the research planning, design, collaboration with participants, and data collection and analysis. In addition, copies of all relevant documents (contract documents, material distributed to the teachers, information about the project given to the participating students and their parents, coding protocols, etc.) and all data (the project log, field notes, video recordings, audio recordings, questionnaire responses, student products, and transcriptions) are securely stored and will be saved for five years after the completion of the project, after which they will be deleted. The participating school, the participating teachers and I have signed a contract (Appendix A) consenting to participation in the project and data collection for the purpose of research. The students and their parents have been informed of the project and the data collection (both in writing and at a physical meeting). All students and parents have been offered the possibility of declining participation in video recordings and interviews.

In addition to these formal ethical considerations, there are certain ethical issues related to doing research that involves children. There is an implicit power relationship between an adult and a child that might make the child feel that (s)he is obligated to participate in, for instance, an interview, or that (s)he is expected to give certain answers or behave in a certain way (Kampmann et al., 2017). A child may also feel uncomfortable, for example, in an interview situation or during an observation. Hence, it has been an important focus to make the students in this study feel safe and comfortable. They have all been carefully informed of the purpose of my presence in the classroom, the aim of the study, and the data administration that includes safe storage of any information that may be linked to them. Furthermore, I have reassured them that no one besides me will have access to recordings or field notes. Especially during the interviews, I have paid special attention to informing them that I would not share information about them or their statements with their teacher or their parents, and that all data will be anonymized or pseudomized. On the one hand, I seek to make the students comfortable in the interview situations, but I also create a safe space for them to share their honest thoughts without worrying about what they might be expected to think. Part of the interview strategy has been to give room for the students to talk about other topics than mathematics (e.g., hobbies, vacations, family, etc.) to make the conversation less formal, which might make the situation more relaxed while also encouraging them to be more honest. Hence, the purpose is both related to ethical considerations and to methodical concerns. Still, I have on the other hand been careful not to pressure the students with my questioning, making them answer questions that they do not wish to answer. In the introduction to the interviews, I have explicitly expressed that they are welcome to decline answering at any time during the interview. I have, furthermore, been very attentive to the students' willingness to participate in interviews. A few students, who might have made interesting cases judged from observations, have not been selected, as they did not seem comfortable being interviewed.

The four focus students have not been informed of the special attention that is given to their development. As will be described in section 4.3, twelve students were interviewed in the beginning of the intervention, and these twelve students were told that I might interview them again during the two years of intervention. However, they were *not* told that some of them would be selected for special focus, for two reasons. First, it might make them wonder why they were or were not selected, thereby increasing their self-awareness in the mathematics lessons and perhaps making them modify their behavior, which ultimately would bias the data. Second, some students might not be bothered by the special attention or by being filmed during mathematics lessons, while others might find it quite disturbing or even unpleasant. I have thus primarily recorded the whole classroom and made sure to vary which students I filmed during group work, though still attempting to focus on at least one of the focus students.

Another ethical issue lies in the extensive collaboration with the teachers, which requires considerations about the distribution of roles and the allocation of expertise. As mentioned in section 3.2.1, an essential feature of a DBR approach is its collaborative nature, where the teachers are co-participants and not merely subjects. It is central to the implementation of the intervention that the teachers have a sense of agency in the intervention and that they contribute with expertise from their perspective of practice. They are the ones who are responsible for realizing the intention of the study, facilitating the development of the students' beliefs. Their ideas, expectations, ambitions, and objections are essential, as they contribute to the study with a perspective that I am not able to have. I do not know the students, the culture, and the norms of the classrooms as they do, and I did not plan the teaching in detail. As a researcher, I am thus primarily responsible for theoretical contributions. To accommodate any potential conflicts of interest or contrasting aims and ambitions, this distribution was articulated and discussed before the intervention.

Chapter 4: Method

The philosophical worldview, the strategy for inquiry and the quality criteria all constitute the foundation for the third major element in the framework presented in figure 12 at the beginning of chapter 3: the research methods. These involve the forms of data collection, analysis and interpretation used in the study (Creswell, 2009). In this chapter, I elaborate on the applied methods and their relations to theory, methodology, and practice.

4.1. Data collection and triangulation

As described in chapter 2, beliefs can be either conscious or unconscious, and they are expressed through actions and utterances. As part of the trustworthiness of the study, and to ensure maximum coverage of the students beliefs in the data collection, several types of data are triangulated (figure 15), covering as much of the spectrum as possible (Bakker & van Eerde, 2015; Creswell, 2009; Schoenfeld, 2007a).

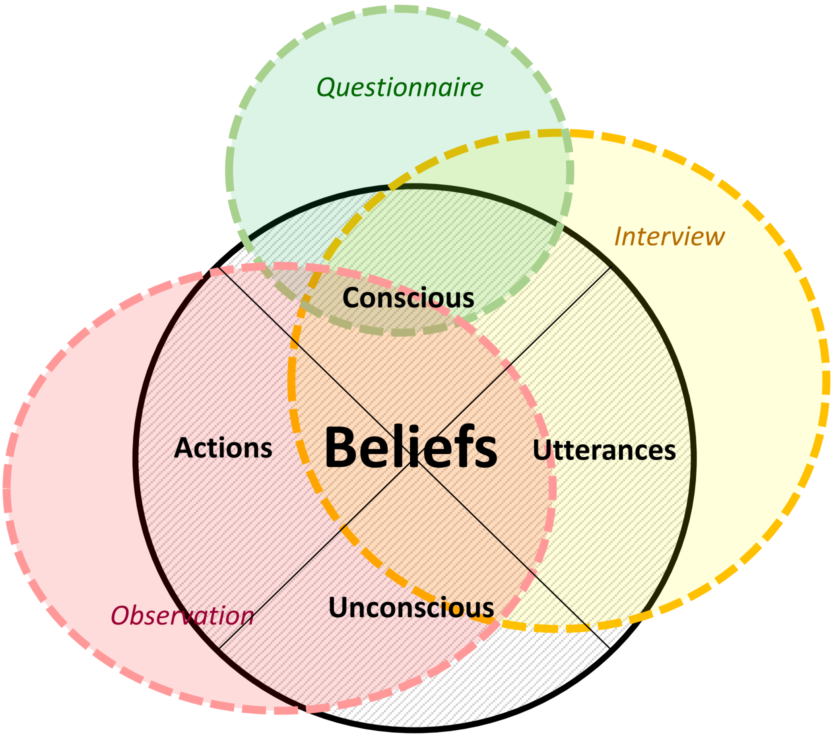


Figure 15: Triangulation of data types

First, a *questionnaire* is answered by all students in the two participating classes (N=43 in 2019, N=38 in 2021). A questionnaire makes it possible to receive answers from a large group of students, but there are certain limitations to this type of data collection. The questions must for example be clear, short and easy to read, and there is no possibility for an elaboration of the answers. Thus, complex issues must be simplified or left out. Furthermore, it must be possible to complete the questionnaire in a foreseeable time, which restricts the number of questions. There is also a limit of space for the answers, and it is not unthinkable that students in 6th grade will provide short and perhaps superficial answers. A questionnaire, therefore, only uncovers a small part of the students' beliefs, restricted to primarily conscious beliefs. Perhaps some of their statements can be interpreted to reveal more unconscious beliefs. However, since there is no possibility for an elaboration of the answers, such interpretations become quite uncertain. The questionnaire provides an overall, subjective snapshot of the students' beliefs on that particular day.

Hence, a deeper understanding of the students' beliefs requires a possibility for elaboration. This possibility occurs in subsequent semi-structured *interviews* (Kvale, 1997), where the students' responses to the questionnaire can be addressed and further investigated. It enables an interpretation of the students' statements, which might uncover some of the more unconscious beliefs. Moreover, it is possible to address more complex matters. However, the interview also has its limitations. It is quite time-consuming, which limits the number of students. And like a questionnaire, an interview is based on self-reporting and can only include the students' stated thoughts and reflections that are their utterances of beliefs.

Beliefs that are expressed through actions might be uncovered through questions about typical or hypothetical behavior, but only through *observations* can such actions be studied and analyzed for indications of both conscious and unconscious beliefs. Therefore, during the entire intervention period, the teaching is observed and video-recorded approximately once every two weeks with specific focus on the selected students. The students' actions and language in mathematical situations are studied in these observations and might provide data about the students' unconscious beliefs, as well as beliefs expressed in actions. Furthermore, the observations document how the principles are realized in practice and how this is related to the intended design.

The following sections present the design, implementation, and analyzing strategy of these three types of data.

4.2. Questionnaires

4.2.1. Design and implementation of pre-questionnaire (2019)

43 students responded to the questionnaire in September 2019, 21 of them in class X, and 22 in class Y. It was distributed on paper in a mathematics lesson, where I was present. I introduced the questionnaire to the students, emphasizing the importance of answering without considering what might be "correct", and that their answers would only be read by me. The students were not given a time limit.

The questionnaire can be found in Appendix B. I designed it based on considerations involving existing research, theoretical constructs, and practical aspects. These considerations are described in the following.

Several researchers have used questionnaires to assess students' beliefs about mathematics. As presented in chapter 2, Jankvist (2015b) has worked with the exposing of upper secondary students' overview and judgment, including a questionnaire with open-ended questions. This questionnaire is structured in three parts, classified by the three forms of overview and judgment—a structure which is adopted in the questionnaire of the present study. Likewise, several of the questions in Jankvist's questionnaire have served as inspiration, which will appear in the presentation of each question below. Also, some of Jankvist's questions are used to characterize the belief dimension of mathematics as a discipline (see section 2.2.3) and have been included in the questionnaire.

Other studies have made use of a questionnaire when seeking to uncover students' beliefs about mathematics. In his exploration of upper secondary students' mathematical beliefs, Schoenfeld (1989) primarily used statements with multiple-choice answers, but he also included a few open-ended questions that allowed students to give more extensive answers. Markovits and Forgasz (2017) investigated mathematics-related beliefs of elementary school students using a mix of open, half-open and closed questions. Likewise, Cheeseman and Mornane (2014) presented students from the same age group with statements to scale along with a few open response questions. In the small-scale study "Kids talk about math", Grootenboer and Marshman (2016) used a small-range questionnaire to investigate the beliefs and attitudes among 9-12 year old students, finding a common perception of mathematics being important, but primarily related to numbers and arithmetic. A few of the questions have served as inspiration for the questionnaire in this study.

As mentioned in section 2.1.5, Johansson and Sumpter (2010) mix three forms of questions in their questionnaire directed towards elementary school students: statements with a Likert-scale, open-ended questions and a drawing task. The questionnaire in this study, likewise, mixes different questioning formats, with a majority of open questions, one multiple-choice question and two rating-scale questions, which will be elaborated below. Similar to Johansson and Sumpter, the questionnaire also includes a drawing task, asking the students to draw themselves in a situation doing mathematics. As also described in section 2.1.5, drawing can be a way to access implicit beliefs, in particular for students who have difficulties with expressing their beliefs in a written questionnaire.

This study's questionnaire is partly structured on the three forms of overview and judgment, as found in Jankvist (2015b), with the addition of introductory questions concerning the students' attitudes toward and beliefs about mathematics as a school subject (questions 1-5), as well as a drawing task concerning mathematical activity and their beliefs about themselves as learners of mathematics, inspired by Johansson and Sumpter (2010) (questions D1 and D2). The questions related to OJ1 (questions A1-A5) include aspects about the importance of learning mathematics and the use of mathematics, both in the students' own life and in society. Questions concerning OJ2 (questions B1-B4) mainly revolve around the origin of mathematics, while OJ3 is represented through questions (questions C1-C4) about mathematical methods, the special character of mathematics compared to other subjects, and the students' perceived criteria for success in mathematics.

The questionnaire consists of 20 items, whereof two are closed questions with a scale format answer ("How well do you like math in school?" and "Do you think you are good at math?"), one is a multiple choice ("What is the most important in math? Choose max. 3 things"), and one is a drawing task ("Draw yourself working with mathematics that you find fun"). The remaining 16 items are open-ended or open response questions. Mixing different types of questions (as e.g., Johansson & Sumpter, 2010; Markovits & Forgasz, 2017), is to begin with an attempt to reach as many students as possible. Some of them will find it challenging to formulate their thoughts about mathematics in words and some will find scales and multiple choice narrow and limiting (Cohen et al., 2011). Also, while some students express themselves best with words, others can show their reflections in a drawing more easily.

Furthermore, the different question types make it possible to reach different forms of beliefs and different levels of reflection. The open response questions (e.g., "How do you imagine that the math you learn at school, has come into being?") aim to access the students immediate and unrestricted thoughts about a certain aspect without limiting them to an already constructed formulation. This allows them to reflect freely, which may provide access to their actual beliefs instead of the beliefs they perceive as the "correct" ones. The open-ended questions (e.g., "Do you think it is important for you to learn math? Yes, because… / No, because…") give the students the beginning of a sentence, which they must finish. The idea is that this leads the students' thinking in a certain direction, presenting an image of their beliefs about a specific aspect of mathematics. Closed questions may be even more restraining, since there are a limited number of response options. For example, the question "What is most important in math? Choose max. 3 things" forces the students to take a stand. This may possibly reveal some of their unconscious and internal beliefs because some of the response options might extend beyond their own associations and ideas.

In the following, I present the considerations behind each question.

Questions 1-5: Attitude towards mathematics

The first 5 questions serve two purposes: to introduce the subject in an accessible way and to assess the students' general relationship with mathematics, primarily as a school subject. Following the attitude framework described in section 2.1.1, the questions address the students' emotional disposition, their vision of mathematics, and their perceived competence.

Question 1. *What do you think of when you hear the word "mathematics"?*
The first question in the questionnaire is somewhat inspired by Grevholm (2011), who posed the question "What is mathematics?" in her study of upper secondary school student views of mathematics. The question directs the students' thinking towards mathematics, which is the overall subject of the questionnaire. In addition, the formulation "what do you think of" invites the students to express their personal associations with the word mathematics, i.e., their *vision* of mathematics. Thereby, answers to this question can reveal if a student, for example, relates mathematics to school or daily life, to science, to certain emotions, or perhaps to a certain content or situation. The question is open to give room for all aspects of associations.

Question 2. *How much do you like math in school?*
This question is included to get a sense of the students' emotional disposition towards mathematics as a school subject. Their level of enjoyment and motivation could possibly affect their beliefs about mathematics in other settings. This question is answered by marking one of four options on a rating scale, spanning from a quite unhappy to a very happy emoticon. The equal number of options is intentional, as it excludes a neutral "middle" option, thereby encouraging the students to take a subjective stand. Furthermore, there are only a few options to make the choice as simple as possible.

Question 3. *Do you think you are good at math?*
The same rating scale is used in question 3, which assesses the students' own perception of performance level, i.e., their perceived competence. The phrasing "Do you think…" clarifies that it is the student's personal opinion that must be indicated, not the opinion of, for instance, the teacher, parents, or classmates. Answers to this question might give an indication of the students' self-efficacy as well as their beliefs about themselves as learners of mathematics.

Question 4. *What do you like the most about math?*
Question 4 is an open question, giving room for various kinds of answers that might be related to specific content (e.g., "fractions" or "geometry"), activities (e.g., "projects", "math games" or "doing calculations"), organizational characteristics (e.g., "group work"), or other aspects (e.g., "my teacher"). The students' answers to this question give an idea of which aspects of mathematics are related to their motivation and may provide information on their emotional disposition, as well as their vision of mathematics. Both question 4 and 5 about what the students like and dislike are inspired by similar questions in the "Kids talk about math" study (Grootenboer & Marshman, 2016).

Question 5. *What do you like the least about math?*

As with the previous question, there is room for a variety of answers, indicating which aspects of mathematics do not bring motivation.

Questions A1-A5: The application of mathematics

As mentioned, the five "A"-questions concern OJ1, the actual application of mathematics. The questions must capture both the personal application of mathematics in daily life and the role of mathematics in society. Furthermore, the importance and relevance of the subject are included in these questions. All five questions are rephrasings of questions found in Jankvist (2015b).

Question A1. *Do you think it is important for you to learn math? Yes, because: / No, because:*

Question A1 is related to the students' feeling of personal relevance when learning mathematics. The phrasing "do you think" emphasizes the subjective character of the questionnaire. There are two options for answering: yes or no. These are both followed by an invitation to elaborate by continuing the sentence "because…". In this way, the students must not only decide whether or not they find mathematics important for their own life, but also give a reason for this. Hence, this might give an indication of their beliefs about what mathematics can be used for, for them personally outside the world of mathematics itself.

Question A2. *Do you think it is important for everybody in Denmark to learn math? Yes, because: / No, because:*

The same considerations apply to the phrasing of this question, but the issue here is the general relevance of learning mathematics. Some students might not be able to see the subject's personal relevance, but still recognize that it can be useful for people in general, or vice versa. As in the previous question, the element of elaborating the answer can both give an idea of the students' beliefs about what mathematics is used for, and of their ability to exemplify their beliefs as well as reflect on the fact that everybody must learn it in school.

Question A3. *What do you experience and think that math is used for in your daily life?*

In contrast to question A1, this question does not concern relevance. Here, the students are asked about their personal experiences with the use of mathematics outside school, and also about what they imagine it is used for. The phrasing "what do you experience and think" thus indicates that there are no wrong or right answers. The students can write whatever comes to mind without worrying about it being correct. The question is open. This gives the students an opportunity to answer as much or little as they find suitable, as well as in whichever direction their associations take them. Hence, their answers might give an impression of their beliefs about the application of mathematics in their own lives, and the experiences and reflection that forms the basis for these.

Question A4. *What do you experience and think that math is used for in society?*
Similar considerations apply to this question, only regarding society instead of personal life. It might be noted in retrospect that several of the subsequently interviewed students struggled with understanding the word "society", which makes the trustworthiness of the answers to this question doubtful.

Question A5. *Do you think that mathematics is more important now than 100 years ago? Why/why not?*
Although this question is placed within the category of OJ1, it is also related to OJ2 and thus combines these two aspects of mathematics. However, the question is primarily designed to make the students reflect on the challenges connected to the present society that might involve the application of mathematics. The intention is that the question might start their associations about mechanisms in our society, such as digitalization etc. The number of 100 years is chosen to signal a society quite different from the present, but not so long ago that some students might not imagine that mathematics did not exist. Again, the phrasing indicates subjectivity and highlights the importance of the reflections connected to their answers.

Questions B1-B4: The historical development of mathematics

The four B-questions are related to OJ2, the historical development of mathematics. This is of course an aspect too comprehensive to capture in just four questions, so the issue in these questions has been narrowed down to concern different aspects of the origin of mathematics. The hypothesis is that such reflections might be transferable to the students' general beliefs about the historical development of mathematics. If a student, for example, imagines mathematics as something static that has been discovered, he or she might not think that mathematics has changed or developed significantly over time. With this particular focus, the questions also leave room for philosophical considerations. Hence, they are all open questions that do not limit the students' responses. Questions B1-B3 are inspired by questions in (Jankvist, 2015b).

Question B1. *How do you imagine that the math you learn at school has come into being?*
The origin of mathematics can be a very abstract issue for students in 6th grade. Therefore, the first question concerning the historical development of mathematics relates to mathematics that the students are familiar with, namely the mathematics they meet in school. In this question, they are asked to use their imagination to consider where this might stem from. The answers to this question might reveal their beliefs about the origin of mathematics, but also their beliefs about what kind of mathematics belongs in school. Compared with the other B-questions, it might give an indication of whether they perceive school mathematics as similar to mathematics in general.

Question B2. *Why do you think somebody came up with mathematics?*

Here, the students are asked to reflect on possible reasons for why mathematics exists. The intention behind the phrasing is to simplify a rather complex question and to give the students an opportunity to use their imagination based on their beliefs about both history and the application of mathematics.

Question B3. *When do you think mathematics came in to being? (specify year or period)*

The third question in this category is intended to reveal the students' sense of time in relation to mathematics. The remark in the parenthesis is meant to support the students in their answers, clarifying what kind of answer is desired. However, this remark might have been the reason for what turned out to be some very specific time indications (which will be elaborated in the data analysis). Retrospectively, the students' answers might have been more in line with their beliefs without the remark, which seems to have limited the character of the responses.

Question B4. *Did you ever learn anything in school about how mathematics came in to being? No / Yes. What was it about?*

To make the students search their memory for previous experiences connected to the historical development of mathematics, the final question in the category asks them to indicate whether they heard about the origin of mathematics in school and, if so, what this might be. The intention of the question is mainly to get an indication of the extent of experiences on which their beliefs might be based.

Questions C1-C4: The nature of mathematics as a subject area

The C-questions in the questionnaire concern OJ3 and primarily address the characteristics of mathematics, both as a school subject and a discipline.

Question C1. *What is the difference between math and other school subjects?*

To encourage the students to reflect on which characteristics might be specific to mathematics, they are asked to compare it to other school subjects, which puts a quite diffuse question in a concrete and relatable context. Thereby, the students might be able to distinguish certain features that are special to mathematics. However, their answers are of course dependent on their experiences in the classroom and their ability to see the school subject in a meta-perspective. Hence, information on these aspects is also part of the intention of this question.

Question C2. *What is the most important in math? Choose max. 3 things:*

a. To get a correct result

b. To be able to remember

c. To use the correct methods

d. To come up with your own methods for solution

e. To understand what the teacher means *f. To be able to explain what you think*

g. To solve problems

h. To know the multiplication tables

i. To be able to find patterns

j. To get good ideas

k. Other: ________________________________

The "point" of mathematics is the issue of this question, i.e., the goals and success criteria that the students associate with mathematics, may it be related to school mathematics or mathematics as a discipline. It is also possible that parts of students' beliefs about the character of mathematical activity will be revealed. This question is the only multiple-choice item in the questionnaire. The considerations behind this choice is that it might be too diffuse for the students, if they were to come up with different aspects, goals or success criteria on their own. The pre-formulated options might include aspects that they would not have thought of themselves. On the other hand, the options might limit their ideas of what can be the answer to this question, which is the reason for not only the open option (k) that gives the students the opportunity to write their own suggestion, but also for the relatively extensive number of options. These include aspects of mathematics from both ends of the dualistic/relativistic spectrum, some clearly related to school mathematics (e.g., option e) and some more general (e.g., option j).

<u>Question C3.</u> *What is a mathematical problem?*

This question seeks to address the characteristics of mathematical activity and methods. It could have referred to, for example, mathematical reasoning or mathematical proofs, but these concepts might be rather unknown to students in 6th grade, where there is a better chance of them having heard of a "mathematical problem" or even have experiences with working with one.

<u>Question C4.</u> *What do you think a mathematician does?*

Here, the students are again asked about their opinion or what they imagine. The question concerns the character of mathematics as a science as well as mathematical methods. A mathematician is thus a representative for the scientific discipline of mathematics, but also for the application of mathematics in society or on the job market. The question is open, making it possible to answer whatever comes into mind.

Questions D1-D2: Drawing

<u>Question D1.</u> *Draw yourself working with mathematics that you find fun*:

The last two questions in the questionnaire are related to a drawing task, with a large square printed on the paper to do the drawing in. The intention with this task is to uncover the students' beliefs about mathematical working methods and their personal relationship with mathematics. In the first draft of the questionnaire, the task was just to "draw yourself working with mathematics", as in Johansson and Sumpter (2010). However, there is a risk that this phrasing would have resulted in a comprehensive number of drawings showing students sitting by a desk, working on a mathematical task, as found in the study by Johansson and Sumpter. Thus, the last part of the sentence ("that you find fun") is an attempt to counter this, as the students with this addition are encouraged to think of an activity that exceeds other experiences in enjoyment.

<u>Question D2.</u> *Explain your drawing:*

Both Rolka and Halverscheid (2011) and Aguilar et al. (2016) have successfully asked the students in their studies to explain their drawings in writing. Some students might not feel that they are able to express themselves adequately through a drawing. The written explanation makes it possible for them to support or clarify their message, further enabling me to distinguish the motive in cases of doubt. In addition, it makes it possible to see what the students perceive as the primary focus of the drawing.

4.2.2. Strategy for analysis of pre-questionnaire

The students' responses to the questionnaire are coded, following a protocol that describes coding criteria for each question. This protocol is inductively designed, based on a scanning of the students' answers, giving an idea of which types of answers they give. Each question has its own purpose and value and must therefore be coded individually (examples are given below). However, there is a general pattern in the coding. As one of the purposes with the questionnaire is to provide information for selecting focus students, and since I seek to select students that represent a variety of profiles, it is necessary to be able to distinguish their level of reflection and their general attitude towards mathematics. Hence, the coding is generally designed to assess these features on a discrete scale from 1 to 4, with 1 representing a low level of reflection and/or negative attitude and 4 representing a high level of reflection and/or positive attitude. The value "0" is assigned to missing or misunderstood responses. In this way, the sum of the coding can give an idea of the students' individual reflection level and attitude. Furthermore, it can provide information on potential tendencies within each question (e.g., that the majority of the students give a category 3-answer to a certain question). The coding protocol can be found in Appendix C. Here, I shall give a few examples of coding criteria:

<u>Question 1:</u> *What do you think of when you hear the word "mathematics"?*

There are many different ways to answer this question. The first scanning of the students' responses revealed that the students' answers would typically either be emotional or about mathematical content, such as "addition" or "fractions". To define the criteria for the five coding categories to this question using the general principle of increasing the level of reflection and positive attitude, the extreme values of 1 and 4 represents each their end of emotional expression; 1 being negative (e.g., "it is difficult", "I cannot do it", or "oh no") and 4 being positive (e.g., "fun" or "challenging"). The values of 2 and 3 represent answers related to mathematical content, with 2 being specific content (e.g., "multiplication" or "fractions"), and 3 being about more general content areas (e.g., "numbers", "calculating", or "my teacher"). This distinguishing is made on the assumption, that there is a slightly higher level of meta-thinking in category 3, but with the awareness that this might not be the case. Thus, these two categories are not particularly relevant when it comes to level of reflection or attitude, but might be interesting when looking at the overall tendency of answers to this question. Table 2 gives an overview of the coding categories.

Table 2: Coding categories, question 1.

0	**1**	**2**	**3**	**4**
No answer / misunderstood ("nothing")	Emotionally negative ("it is difficult", "I cannot do it", "oh God, not that again", "boo", "oh no"	Content, specific ("multiplication", "fractions")	Content, general ("numbers", "calculating", "my teacher", "calculations")	Emotionally positive ("fun", "challenging")

Question A2: *Do you think it is important for everybody in Denmark to learn math? Yes, because: / No, because:*

The first scanning revealed a clear preference for answering "yes" to this question, and three of the coding categories are thus related to this option. The categories 2, 3 and 4 increase in level of specification, which relates to the students' ability to reflect on this matter (see table 3).

Table 3: Coding categories, question A2

0	**1**	**2**	**3**	**4**
No answer	No	Yes Unspecified justification or vaque ("it is important", "you need it")	Yes Generally specified ("to get an education", "to get a job", "you use it for everything")	Yes Further specified ("I'm going to be an engineer")

Question B3: *When do you think mathematics came in to being? (Specify year or period)*

The correctness of the students' answers to this question is not essential. Instead, the range in the coding concerns the students' level of reflection and ability to justify the answer, which is why the highest coding value (4) represents any *justified* time indication, regardless of how correct this might be, and the lowest (1) represents "I don't know". Within this range, an apparently random time period is assigned a higher value (3) than an apparently random year, since an answer that is constituted by a single year is considered quite far from reality and, thereby, shows a low level of reflection.

Table 4: Coding categories, question B3

0	1	2	3	4
No answer / misunderstood	Don't know	Specific (random) year ("1773", "1900")	Period (random) ("the Iron Age")	Justified time indication ("since humans came into being")

Question C2: *What is the most important in math? Choose max. 3 things*

As this question is a multiple-choice item, it is not possible to code the answers by following the general coding principle with four categories, representing an increasing level of reflection and/or positive attitude towards mathematics. Instead, responses to this question are coded using two values: 1 for a marked option and 0 for an unmarked option. In this way, a student's answer to this question does not influence the total sum of the coding values, as all students who have answered it correctly will get the value "3". The question is thus analyzed by investigating if there are certain options that are marked more often than others, or if some are not marked by any students. Coherence between the choices of options has not been analyzed.

Question D1: *Draw yourself working with mathematics that you find fun.*

Interpreting children's drawing is a comprehensive research field that includes complex elements, for example, related to psychology. Much information could potentially be collected from the students' drawings in this questionnaire, for example, concerning the facial expression of persons in the drawings or the placement and role of the teacher in the drawn situations. Keeping the focus of the study in mind, the coding is, however, restricted to information regarding the students' mathematics-related beliefs and their attitudes towards mathematics. Moreover, an extensive analysis of the drawings exceeds the limitations of this study. The coding is thus centered around the situations that the students depict in their drawings, particularly in relation to a school setting (table 5). Hence, a high coding value represents a mathematical situation outside a school-setting, as this indicates that the student's view on mathematics is not restricted to school. In contrast, a lower coding value is assigned to a drawing that depicts what might be referred to as a 'traditional' student role. As some drawings seem to relate to the students' emotions towards mathematics, code 1 is assigned to drawings that clearly express a negativity. As for the coding of other questions, a drawing can be assigned more than one code, for example, if it illustrates a student with a sad face in a traditional student role (code 1 and 2).

Table 5: Coding categories, question D1

0	1	2	3	4
No answer / misunderstood	Emotionally negative	Traditional student role (alone, sitting by a desk, solving tasks)	Other, but in a school setting	Non-school setting

After coding all the responses, the number of answers in each category is counted for each of the two classes (an overview can be found in Appendix D). As an answer can include multiple aspects, the same answer might be included more than once (e.g., in question 4 and 5, where some students mention several aspects of mathematics, or in question 2 and 3, where some students place their mark between the two middle options). For that reason, the total number of answers sometimes exceeds the total number of students.

Due to the format of the questionnaire with a high number of open questions, this form of coding has a quite high risk of uncertainty or bias. There are generally no fixed answering options, and it is up to the coder to assess in which category an answer belongs. Hence, to lower this risk, random questions have been coded independently by two coders (myself and a student employee) comparing and discussing the results. Furthermore, to establish a consistent coding strategy, answers to which there has been doubt about the coding category, have been discussed and assessed by the coder and me.

Each of the question aspects (attitude towards mathematics, application of mathematics, historical development of mathematics, nature of mathematics and mathematical activity) is analyzed separately by searching for tendencies and patterns in the students' responses. At the same time, differences between the two classes are noticed. A clear discrepancy can be a sign that the students' beliefs on a specific area are dependent on class specific factors such as the teacher, classroom norms etc., whereas similar responses are more likely to be generalizable.

Although quantitative methods are applied, the analysis of the questionnaire must in essence be characterized as qualitative. Most of the items are open questions, which are individually interpreted in the coding processes. Also, the numbers of answers in each coding category are objects of interpretation, leading to qualitative findings concerning the overall tendencies in the students' beliefs.

4.2.3. Design and implementation of post-questionnaire (2021)

For the purpose of comparison, the students responded to a new questionnaire in June 2021 after completing the intervention. The total number of students in the two classes were now 38 (19 in each class). The questionnaire was distributed in a mathematics lesson, with both the teacher and me present. As in 2019, there was no time limit for responding, and the students were reminded of the importance of answering truthfully and thoroughly.

The questionnaire distributed to the students at the end of the intervention was more or less similar to the one they responded to in the beginning. Even though the analysis of the pre-questionnaire pointed to some challenges regarding phrasing or content of the questions, these overall remained the same for the purpose of comparing the students' answers in the beginning of the intervention and in the end. However, there were three modifications:

Firstly, the format was changed from paper to a digital questionnaire. This modification was based on two conditions: 1) It would make the data processing easier and more secure in relation to data protection, and 2) one of the students suffered from severe dyslexia and experienced great difficulties when filling out the pre-questionnaire. By making it digital, this student was able to use computer-based support when reading and writing. However, for the drawing task, all students were handed a piece of paper.

Secondly, three of the questions were modified: Question B4 was changed from "Did you ever learn anything in school about how mathematics came in to being? No / Yes. What was it about?" to "What have you learnt in school about the history of mathematics?" As the intervention included content about the history of mathematics, the question should not concern *if* they learned anything, but *what* they learned. The phrasing in question C1 were modified from "What is the difference between math and other school subjects?" to "What do you think is the difference between having math and other subjects?", so that it became clear that the question concerned the students' *opinion* about the matter. Some of the options in question C2 were modified, deleted, or added:

Question C2. *What is the most important in math? Choose max. 3 things:*

a. To get a correct result (unchanged)

b. To know rules and formulas by heart (changed from: *to be able to remember*)

c. To use the correct methods (unchanged)

d. To be able to make decisions (instead of: *to come up with your own methods for solution*)

e. To understand what the teacher ~~means~~ explains (modified)

f. To be able to explain what you think (unchanged)

g. To solve problems (unchanged)

h. To know the multiplication tables (unchanged)

i. To be able to find patterns (unchanged)

j. To get good ideas (unchanged)

k. Other: ________________________________ (unchanged)

Options b and e were changed to make the phrasing more specific and precise. Option d in the pre-questionnaire was replaced with a question that addressed decision making, which played an important part in some of the designed activities in the intervention. It had been a focus point to make the students aware that mathematics can be used to make decisions that are more qualified. In addition to the changes of the phrasing, the options were shown in random order in the post-questionnaire as the digital format allowed this. The randomization prevented any bias connected to the students marking the first options they see.

Thirdly, three questions were added at the end of the questionnaire to evaluate the intervention, as well as investigate which experiences that had made an impression on the students:

Question E1. *What do you remember best from the math lessons in 6th and 7th grade?*

Question E2. *When do you think you have learned most, and why?*

Question E3. *What has been the most fun during math class during the last two years?*

These three questions were not included in the sum of the students' coding values, but the answers might provide information about to what degree the students referred to experiences related to the intervention or not, hence indicating if the intervention had made a difference or impact on their experience with mathematics.

4.2.4. Strategy for analysis of post-questionnaire

The strategy of analyzing the post-questionnaire is the same as the pre-questionnaire (cf. section 4.2.2). Furthermore, the results are compared to the results of the pre-questionnaire, thus giving an overview of possible changes in the students' beliefs and attitudes. The comparative analysis regards the following aspects:

1. The distribution of answers in the five coding categories to each question.
2. The average coding value in each aspect category (attitude, application, historical development, nature of mathematics, mathematical activity, and evaluation of the intervention).
3. The distribution of students in the three color categories, based on the sum of the coding values.

1. and 2. are included in the analysis in sections 8.1.1–8.1.6 concerning each aspect category, while 3. is elaborated in section 8.1.7 in relation to the overall development of the students' beliefs.

The answers to the three added questions concerning evaluation of the intervention are coded according to their level of specification and whether the students mention examples from the intervention. The coding criteria are the same for all three questions and can be found in table 6:

Table 6: Coding categories, questions E1-E3

0	1	2	3	4
No answer / misunderstood	"Don't know" / "can't remember" / "nothing"	Not specified or not related to mathematics	Specified, not related to intervention	Specified, related to intervention

4.3. Selection of focus students

The students' responses to the pre-questionnaire formed the foundation for the selection of students to be interviewed and followed throughout the intervention. As the intervention was aimed at all types of students, the selected students represent a variety of beliefs and attitudes towards mathematics, thus constituting what Flyvbjerg (2006) terms *maximum variety cases*. Furthermore, the selected group of students includes an equal number of girls and boys, as well as an equal number of students from each class. The selection of focus students was primarily done based on thorough readings of their answers, followed by a temporary division in three categories: red, yellow and green, reflecting a spectrum with a negative attitude and/or unreflected beliefs in the red end and positive attitude and/or reflected beliefs in the green end. To support this categorization, the coding values related to responses were, when possible, organized so that a low score indicates a negative attitude or little or no reflection in the answers. Thereby, the three categories are reflected in the sum of the coding values, as seen in figure 16:

Class X, code values (students)

Nr.	Name	Class	Sum 2019	1-19	2-19	3-19	4-19	5-19	A1-1	A2-1	A3-1	A4-1	A5-1	B1-1	B2-1	B3-1	B4-1
101	[redacted]	X	37	0	2	1	1	4	3	3	3	1	4	1	2	0	(
102	**Molly**	X	**56**	**2,5**	**3**	**3**	**4**	**4**	**3**	**4**	**4**	**2**	**2**	**3**	**3**	**3**	**4**
103	[redacted]	X	49	2	3	3	3	4	3	3	3	4	1	3	3	2	1
104	[redacted]	X	40	2	3	3	1	1	3	3	1	1	1	3	3	2	1
105	[redacted]	X	35,5	2	2,5	3	0	0	3	3	3	1	4	1	1	1	4
106	[redacted]	X	40	0,5	2	2	3	3	3	3	4	3	1	2	2	2	1
107	[redacted]	X	46	3	3	1	4	4	3	3	4	2	1	3	1	2	4
108	[redacted]	X	54	3	3	3	4	4	3	3	4	2	4	3	4	2	4

Figure 16: Example of categorizing according to sum of coding values. Red: <41, yellow: 41-49, green: >49. Excerpt of data analysis.

Table 7: Categorizing of all students according to the sum of coding values. Red: <41, yellow: 41-49, green: >49.

Class X

Name	Class	Sum 2019
[redacted]	X	37
Molly	X	**56**
[redacted]	X	49
[redacted]	X	40
[redacted]	X	35,5
[redacted]	X	40
[redacted]	X	46
[redacted]	X	54
[redacted]	X	42
[redacted]	X	43
[redacted]	X	19,5
[redacted]	X	56
[redacted]	X	50
Tom	X	**47**
[redacted]	X	57
[redacted]	X	46
[redacted]	X	42,5
[redacted]	X	52
[redacted]	X	32
[redacted]	X	42
[redacted]	X	30
[redacted]	X	
[redacted]	X	
[redacted]	X	
Average		43,64

Class Y

Name	Class	Sum 2019
[redacted]	Y	34
[redacted]	Y	49
[redacted]	Y	47
[redacted]	Y	54
[redacted]	Y	48
Adam	Y	**46,5**
[redacted]	Y	47
[redacted]	Y	55
[redacted]	Y	51,5
[redacted]	Y	45
[redacted]	Y	36
[redacted]	Y	50
[redacted]	Y	49
[redacted]	Y	50
[redacted]	Y	41
Erica	Y	**32**
[redacted]	Y	44
[redacted]	Y	36
[redacted]	Y	42
[redacted]	Y	47,5
[redacted]	Y	40
[redacted]	Y	52
[redacted]	Y	
[redacted]	Y	
Average		45,30

In each of the two classes, I selected two students from each category. However, one student (Tom) was included because of his very atypical profile, with a mix of the red and the green category, as his answers indicate a negative attitude, but a reflected answer. Tom thus represents a so-called *deviant case* (Flyvbjerg, 2006), providing information of unusual examples.

I thus presented a suggestion of twelve students to the teachers, and we discussed if some of the students might not be suitable, for example, because of severe learning difficulties in all subjects or because of personal matters. This discussion provided me with essential information, which made me able to make the final decision. Furthermore, I interviewed all twelve students to form an impression of their ability and willingness to participate and share their thoughts. I thus ended up selecting six students. Unfortunately, two of them left the school during the first year, and were naturally excluded from the study. The remaining four students that form the cases below are listed in table 8. The analysis of their beliefs in section 5.2 includes detailed descriptions of the four students along with an individual elaboration of the reasons for selecting them.

Table 8: Overview of focus students

Pseudonym	**Class**	**Gender**	**Sum of coding values**	**Color category in questionnaire**
Tom	X	M	47	Yellow (note: mix of red and green)
Molly	X	F	57	Green
Erica	Y	F	32	Red
Adam	Y	M	46.5	Yellow

4.4. Interviews

Interviewing the selected focus students provides an opportunity to obtain a deeper understanding of their beliefs, not only by having the students elaborate on their answers to the questions in the questionnaire, but also by asking them about related issues, such as their learning behavior and other dimensions of their mathematics-related belief system. These issues can contribute to a more nuanced image of the students' beliefs about mathematics as a discipline and give access to both conscious and unconscious beliefs. A student's statements on, for example, their typical learning behavior might reflect his or her beliefs about the character and purpose of mathematical activity, about the learning of mathematics or about mathematics education.

4.4.1. Design and conduct of pre-interviews (2019)

The interviews were conducted using a semi-structured approach (Kvale, 1997). This means that the interview guide is prepared from theoretical knowledge and includes questions that I wished to explore. However, the questions are to some extent meant as a guide and can be altered, skipped, or elaborated on during the interview. Thereby, the interview can be adapted to the individual student, making it possible to pursue interesting aspects or to omit parts that appear irrelevant in the situation. More importantly, it enables the students to reflect and associate, thereby providing access to expressions of their beliefs.

The interview guide for the first round of interviews (Appendix E), includes the following headlines: Attitude towards mathematics, learning behavior, beliefs about the self, beliefs about the applications of mathematics, beliefs about the historical development of mathematics, beliefs about the nature of mathematics as a subject area, and mathematical activity. However, the dimensions of the belief system concerning mathematics education and the classroom context are indirectly addressed in some of the questions. For example, the question "What do you do if there is a task you cannot solve?" not only addresses the student' behavior in an individual learning situation, it also provides information on the norms in the classroom and the students' beliefs about the expectations for, and role of, students and teacher (i.e., the dimension concerning beliefs about the classroom context, cf. figure 4, section 2.2.1).

The structure of the interview guide is based on three types of questions: research questions, describing the purpose of asking the questions; interview questions, to be asked in the approximate phrasing; and follow-up questions, which are ideas for questions that might be relevant to include, depending on the direction the conversation takes. In addition, the guide has a column for individual notes for relevant information from the interviewed student's questionnaire response as well as for notes taken during the interview if necessary. The questions are generally designed to initiate the students' reflections without leading them in the direction of a certain answer. In the introduction to every interview, it was made clear that honesty and spontaneity were important and preferable, compared to answers that the students might imagine to be expected of them.

In essence, the interview guide works as inspiration and checklist for the conversation. As described in relation to ethical considerations in section 3.3.4, it is essential to establish a comfortable and safe atmosphere when interviewing children, in order to make them feel relaxed and willing to express their thoughts. This was especially important in the pre-interviews, where the interpersonal relationships between the individual students and myself were established. Therefore, the conducted interviews included a certain amount of what might seem to be irrelevant small talk concerning issues outside the topic of mathematics, for example, other school subjects or the students' leisure activities. The average length of an interview was approximately 30 minutes. The interviews were conducted in a separate, quiet room, so that the students did not have to worry about others listening. Furthermore, as mentioned earlier, they were ensured that the recordings would be used for research purposes only and would not be shared with anyone.

4.4.2. Strategy for analysis of pre-interviews

The analysis of the interviews begins with the choices made in connection to the *transcription*. In this study, the interviews were transcribed using *intelligent verbatim* transcription method (McMullin, 2021), where every word and element in the conversation is written down, including hesitations, laughter and fillers such as “uh” or “you know”, where it seems relevant. These elements might provide information about the students’ reflection processes or uncertainties. If a student, for example, is very hesitant or speaks in broken sentences, it might be a sign of doubt, of guessing or that he or she has not previously thought about the specific topic. However, irrelevant elements were omitted from the transcriptions (e.g., intonation, grammatical errors, or irrelevant cases of pausing, laughing, etc.). Clarifying comments were added in [brackets], and unclear audio was marked as (-ua-). The transcriptions include line numbers and markings of time every full minute. Figure 17 shows excerpts of a transcription template.

No.	Time (min.)	Speaker	Audio
1	00.00	Interviewer:	
2		Student:	
3		I:	
4		S:	
5	01:00	I:	
6		S:	

Figure 17: Excerpt of transcription template

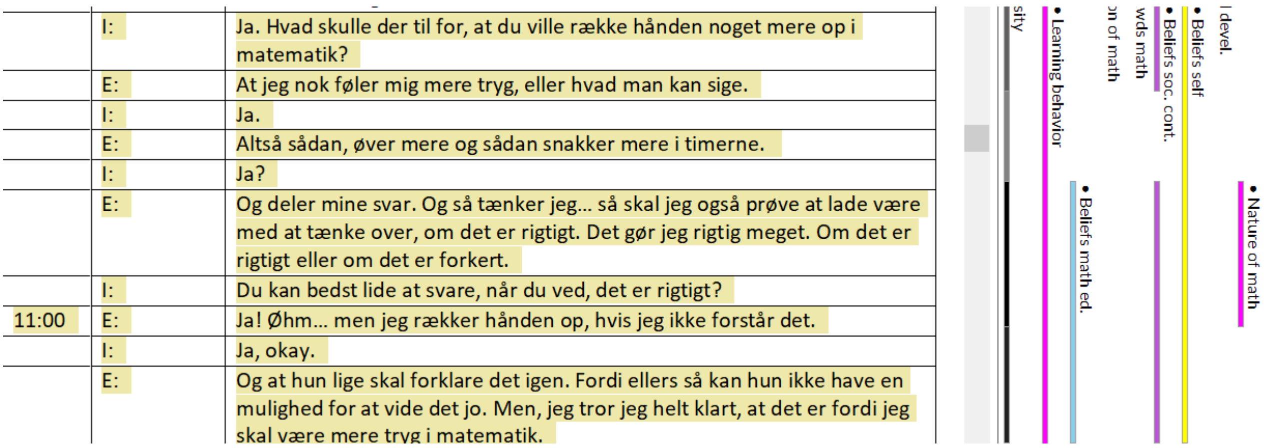
The transcriptions were exported into the software nVivo for coding. Every statement was assessed and assigned one or more of the eight codes listed below according to its content. These codes entail aspects which are thought to possibly influence or relate to the students’ beliefs about mathematics as a discipline, and which were all addressed in the interview. They include the three forms of OJ, the remaining three dimensions of the mathematics-related belief system and learning behavior.

1. **attitude towards mathematics**: statements indicating a student’s emotional or subjective relationship towards mathematics, perceived competence, and vision of mathematics (cf. the framework on attitude by Di Martino and Zan (2010), presented in section 2.1.1), (e.g., “mathematics is boring”, or “mathematics is my favorite subject”). These are statements that might be dependent on the student’s mood that particular day and can change rather easily.
2. **beliefs about application of mathematics**: statements that concern the relevance or importance of mathematics, as well as the use of mathematics. This includes which areas mathematics is used for, how it is used, who is using it and why, and the role of mathematics in the world (e.g., “mathematics is used for shopping”, “my mom uses mathematics in her job as a nurse”, or “without mathematics, society would collapse”)

3. **beliefs about the historical development of mathematics**: statements concerning the history of mathematics as well as the development of mathematics (e.g., "mathematics was invented by the Egyptians", "computers have changed mathematics", or "mathematicians invent new mathematical methods". The latter statement also belongs in code 4, as it indicates the work of a mathematician, i.e., the character of mathematics as a scientific discipline).
4. **beliefs about the nature of mathematics as a subject area**: statements indicating a student's perception of the character of mathematics as a scientific discipline, including mathematical working methods, success criteria, mathematical content areas and epistemological aspects (e.g., "mathematics is about numbers", "in mathematics you need to find the correct result", or "persistence is essential when learning mathematics".
5. **beliefs about mathematics education**: statements concerning the teaching and learning of mathematics (e.g., "we mostly do individual exercises", or "in mathematics you need to finish quickly").
6. **beliefs about self**: statements that describe the students' perception of themselves as learners of mathematics, including self-efficacy, performance level and working habits (e.g., "I find mathematics very easy", "I do not normally participate in classroom discussions", or "I give up easily").
7. **beliefs about social context (the classroom)**: statements concerning norms in the classroom, including both general norms and socio-mathematical norms (e.g., "nobody makes fun of you, if give a wrong answer", "if I am not able to solve a task, I ask the teacher for help", or "we normally work in pairs").
8. **learning behavior**: statements that describe how the student behaves in learning situations, in the classroom, in group work, and in individual situations (e.g., "it is hard for me to concentrate in class", "I spend at least 10 minutes trying to solve a task before seeking help", or "I like challenging tasks").

The last four codes are only applied in the first phase of the coding, as these statements subsequently are used for nuancing the image of the students' attitudes and beliefs concerning the three forms of OJ. The statements coded in categories 5-8 are thus included in the analysis of categories 1-4. Furthermore, they contribute to the overall description of the student. As the examples above illustrate, the codes overlap each other, as for example statements concerning learning behavior might also reflect the student's beliefs about mathematics education or self. In addition, it can sometimes be difficult to distinguish between the coding categories. In such cases, a statement is assigned all relevant codes. For example, the statement "I only raise my hand when I am completely sure that the answer is correct", is in the first phase assigned the codes 4, 5, 6, 7 and 8 as it not only reflects the student's learning behavior in class, but also the student's perception of success criteria in mathematics (correctness is important), the student's self-confidence, the norms in the classroom (a correct answer might be expected or rewarded), and the student's beliefs about mathematics education (mathematical tasks are characterized by a question and an answer, and the teacher knows the correct answer). In the second phase, this statement is included both in the analytical interpretation of the student's beliefs about the nature of mathematics as a subject area as well as in the general description of the student and his or her attitude towards mathematics.

The eight codes are deductively designed, as they partly follow the headlines of the interview guide (attitude, learning behavior, beliefs about self, application, historical development and nature of mathematics, cf. Appendix E), as well as the additional dimensions of students' mathematics-related belief system (beliefs about mathematics education and the classroom context). As the coding process unavoidably is interpretive, there is a risk of subjective bias in the analysis of the interviews. To minimize this risk, the codes and the coding process have been discussed with peers, and a sample of the data has been coded in cooperation with a fellow researcher.



	I:	Ja. Hvad skulle der til for, at du ville række hånden noget mere op i matematik?
	E:	At jeg nok føler mig mere tryg, eller hvad man kan sige.
	I:	Ja.
	E:	Altså sådan, øver mere og sådan snakker mere i timerne.
	I:	Ja?
	E:	Og deler mine svar. Og så tænker jeg... så skal jeg også prøve at lade være med at tænke over, om det er rigtigt. Det gør jeg rigtig meget. Om det er rigtigt eller om det er forkert.
	I:	Du kan bedst lide at svare, når du ved, det er rigtigt?
11:00	E:	Ja! Øhm... men jeg rækker hånden op, hvis jeg ikke forstår det.
	I:	Ja, okay.
	E:	Og at hun lige skal forklare det igen. Fordi ellers så kan hun ikke have en mulighed for at vide det jo. Men, jeg tror jeg helt klart, at det er fordi jeg skal være mere tryg i matematik.

Figure 18: Example of coding in nVivo. The dialogue concerns the student's participation in classroom discussions, which include statements about her learning behavior, her reflections on the importance of feeling safe in the classroom, the character of mathematical answers, and the norms in the class. Hence, the codes on the right show that this dialogue is assigned five different codes: "nature of mathematics", "beliefs self", "beliefs about social context", "beliefs about mathematics education", and "learning behavior". In the subsequent analysis, all these elements are considered, and this statement is included in the interpretation of the student's beliefs about the nature of mathematics as well as in the general description of the student.

After coding a student's statements, they are organized within coding categories, so that, for instance, all statements concerning beliefs about the application of mathematics are displayed together. As some statements do not make sense out of context, most coded statements consist of a dialogue, e.g., the question that precedes the relevant statement or the conversation that is initiated by a certain statement.

The analyses of the interviews are part of an overall analysis of the beliefs of each student, which include both responses to the questionnaire and statements in the interview. In order to analyze the different aspects of the students' beliefs, these responses and statements are organized in a scheme structured according to the categories in the interview. Thereby, a student's answers about, for example, the attitude towards mathematics can be compared to his or her statements on this issue in the interview (figure 19).

Beliefs connected to the three forms of OJ are then analyzed using the theoretical constructs of consistency, exemplification, and justification (cf. section 2.1.5) and include considerations concerning whether the beliefs seem to be held evidentially or not. As mentioned, the data not directly linked to beliefs about mathematics as a discipline (attitude and learning behavior, as well as beliefs about self, mathematics education, and social context) are not explicitly analyzed, but are included for a nuanced profile of the student that may contribute to a deeper understanding of the student and his/her mathematics-related belief system.

Questionnaire	Interview
Attitude towards math	
Oh no, I did not want that Smiley 1 Like best: "I don't know because I cannot stand mathematics" Like least: "everything" "I think it's a bad invention, do not like the subject" Drawing: "me who makes division"	Funny when I can figure it out It can be difficult..., but then, you can easily learn it, you just have to be willing to learn it before you learn it. Has changed attitude: *So it has changed a bit actually?* Yes, I think so too, it has. it is easier the subjects you are good at than mathematics, which probably not everyone is equally good at, or what you can say. Drawing: I definitely think it's because it's probably what I'm best at. Because it's like what I've been practicing a lot and just concentrating a lot on. Funny: 'today's number'. So I just think it was fun. And you could just choose which calculations, for example, you just wanted to calculate. And so, she did it in different ways, many times. And then you had to write a story and such
Summary: *significant change of attitude in three months. Has discovered that she can succeed if she tries.*	

Figure 19: Excerpt of analyzing scheme, presenting answers to the questionnaire on the left and statements from the interview on the right, organized according to topic (Here: attitude towards mathematics).

4.4.3. Midway interviews (2020)

In November 2020, I conducted short interviews with the 12 students that had been interviewed in 2019. The purpose of these midway interviews was mainly evaluative in relation to the intervention.

These interviews are transcribed, but not analyzed as an individual set of data. Instead, they are included in the final analysis as a supplement to the post-data in cases where they provide relevant information about the development of the students' beliefs during the intervention.

4.4.4. Design and conduct of post-interviews (2021)

The design of the interview guide for the post-interviews is very similar to the one for the pre-interviews. It follows the same structure and headlines. The majority of the questions are transferred from the first interview guide, but a few of them have been rephrased, and questions concerning the intervention and the periods of lockdown are added. The interview guide for the post-interviews can be found in Appendix F, where changes are written in red.

Also, the execution of the interviews remained more or less unchanged, except this time only the focus students were interviewed. The length of the interviews was between 20 and 30 minutes, taking place in a separate room.

4.4.5. Strategy for analysis of post-interviews

The analysis of the post-interviews follows the same transcription method and coding strategy as the pre-interviews. This time, however, the interview data are triangulated not only with the post-questionnaire, but also with examples from the classroom observation. Furthermore, the analysis is comparative, as it seeks to investigate a development in the students' beliefs from the beginning to the end of the intervention. It is thus almost a double analysis: one related to the character of each focus student's post-beliefs, and one related to the possible difference between his or her pre- and post-beliefs. Both steps are analyzed according to the dimensions of exemplification, justification, and consistency (cf. section 2.1.5), and the level of evidentiality in the way they appear to be held. Again, the data concerning the students' attitudes towards mathematics, their learning behavior, and their beliefs about the self, the social context, and mathematics education serve as tools for nuancing the analysis of their beliefs about mathematics as a discipline.

4.5. Classroom observations

Approximately two or three times a month, I observed the mathematics lessons (switching between the two classes). All classroom observations were video recorded using one or two cameras (approximately 45 hours of video-recorded teaching). Since the mathematics lessons in the two classes were parallel in the schedule, I was only able to physically attend one of them at a time, so on several occasions, I placed a camera in the class that I was not attending. Where I attended physically, I also took field notes. During the observations, my focus was primarily on two elements: to document the learning activities and realized implementation of the teaching principles, and to capture signs of the selected students' beliefs about mathematics as a discipline in the form of utterances or actions.

Due to the COVID-19 pandemic, the students were attending online teaching from home in two periods: March 2020 – May 2020 and December 2020 – April 2021 (cf. figure 13 in section 3.2). In the first period as well as the rest of that school year, the teachers were only obligated to offer “basic” teaching, i.e., the normal curriculum did not apply as such. In this period, I did not observe the teaching, nor was I in contact with the teachers. In the second period of lockdown caused by the COVID-19 pandemic (December 2020 to April 2021), I was able to attend the online lessons of the two classes, which enabled me to observe almost all lessons. The sound of these lessons has been recorded, and I collected relevant screenshots and took field notes. Approximately 40-50 hours of online teaching have been audio-recorded. Unfortunately, the online format reduced my ability to follow the individual work of the four selected students, as they often logged off during such tasks. On some occasions, I was able to attend some of their group work sessions.

Only selected parts of the data collected from the observations are analyzed, due to the extensive amount. These selected examples of classroom situations serve to illustrate, document, or support important points related to the research questions in combination with data from the other data sources. They have not been transcribed, but simply described in the analyses in chapter 8 as illustrative examples.

Chapter 5: Analysis of students' beliefs about mathematics as a discipline

The analyses presented in this chapter seek to answer RQ1: *What characterizes middle school students' beliefs about mathematics as a discipline?* They thus address the students' beliefs at the beginning of the intervention. The chapter consists of two parts: an analysis of the beliefs of all students based on their responses to the pre-questionnaire (section 5.1), and in-depth analyses of the beliefs of the four focus students, presented as cases (section 5.2).

5.1. Data analysis: pre-beliefs of all students (pre-questionnaire)

In the following sections, I analyze the data connected to each aspect of mathematics: the students' attitude towards school mathematics (questions 1-5), their beliefs about the application of mathematics (questions A1-A5), about the historical development of mathematics (questions B1-B4), about the nature of mathematics as a subject area (questions C1-C4), and about mathematical activity (represented by the drawing, questions D1-D2).

The analysis is based on the data found in table 9, which shows the number of answers in each coding category, both for each of the two classes and in total. The coding criteria for each question can be found in Appendix C, where the number of answers is also displayed. Coding criteria for each aspect of mathematics (attitude, application, etc.) are, furthermore, displayed at the beginning of the respective sections.

Table 9: Number of answers in the five coding categories (1, 2, 3, 4 and 0) in pre-questionnaire. The columns marked X and Y represent the two classes, Class X with 21 students, class Y with 22 students. Some answers belong in two categories and are thus counted in both, which is why the sum of answers in the five categories might exceed the total number of students (N=43).

Questionnaire 2019. Number of answers in categories 1, 2, 3, 4 and 0.															
	Code 1 2019			Code 2 2019			Code 3 2019			Code 4 2019			Code 0 2019		
Question/cat.	1X	1Y	**1 Total**	2X	2Y	**2 Total**	3X	3Y	**3 Tota**	4X	4Y	**4 Total**	0X	0Y	**0 Total**
1. What do you think of, when you hear the word ”mathematics”	5	7	**12**	8	4	**12**	9	10	**19**	1	4	**5**	2	0	**2**
2. How much do you like math in school?	4	1	**5**	6	7	**13**	13	12	**25**	1	5	**6**	0	0	**0**
3. Do you think you are good at math?	3	3	**6**	6	3	**9**	13	10	**23**	1	6	**7**	0	0	**0**
4. What do you like the most about math?	3	0	**3**	4	3	**7**	3	1	**4**	9	16	**25**	2	1	**3**
5. What do you like the least about math?	1	1	**2**	4	4	**8**	4	1	**5**	10	15	**25**	2	1	**3**
A1. Do you think it is important for you to learn math?	2	1	**3**	2	6	**8**	16	11	**27**	1	4	**5**	0	0	**0**
A2. Do you think it is important for everybody in Denmark to lea	3	3	**6**	1	7	**8**	15	10	**25**	2	2	**4**	0	0	**0**
A3. What do you experience and think that math is used for in y	1	2	**3**	1	2	**3**	13	13	**26**	6	5	**11**	0	0	**0**
A4. What do you experience and think that math is used for in s	7	4	**11**	5	6	**11**	7	5	**12**	2	6	**8**	0	1	**1**
A5. Do you think that mathematics is more important now than	6	6	**12**	2	1	**3**	0	6	**6**	8	10	**18**	5	1	**6**
B1. How do you imagine that the math, you learn at school, has	6	7	**13**	2	0	**2**	8	7	**15**	3	4	**7**	2	4	**6**
B2. Why do you think somebody came up with mathematics?	3	0	**3**	4	1	**5**	9	12	**21**	4	5	**9**	1	4	**5**
B3. When do you think mathematics came in to being? (specify y	1	3	**4**	14	14	**28**	3	4	**7**	1	1	**2**	2	0	**2**
B4. Did you ever learn anything in school about how mathemati	9	18	**27**	0	0	**0**	2	1	**3**	7	3	**10**	2	0	**2**
C1. What is the difference between math and other school subje	5	2	**7**	7	5	**12**	9	12	**21**	1	4	**5**	0	0	**0**
C2. What is the most important in math? Choose max. 3 things:															
C2a. To get a correct result	9	6	**15**										12	16	**27**
C2b. To be able to remember	7	1	**8**										14	21	**35**
C2c. To use the correct methods	5	7	**12**										16	15	**31**
C2d. To come up with your own methods for solution	10	12	**22**										11	10	**21**
C2e. To understand what the teacher means	9	8	**17**										12	14	**26**
C2f. To be able to explain what you think	6	10	**16**										15	12	**27**
C2g. To solve problems	4	2	**6**										17	20	**37**
C2h. To know the multiplication tables	8	6	**14**										13	16	**29**
C2i. To be able to find patterns	0	1	**1**										21	21	**42**
C2j. To get good ideas	2	6	**8**										19	16	**35**
C2k. Other	1	3	**4**										20	19	**39**
C3. What is a mathematical problem?	14	13	**27**	3	7	**10**	0	0	**0**	1	1	**2**	3	1	**4**
C4. What do you think a mathematician does?	1	4	**5**	1	1	**2**	12	12	**24**	5	3	**8**	2	2	**2**
D1. Draw yourself working with math that you find funny	2	0	**2**	8	14	**22**	6	5	**11**	3	2	**5**	2	1	**3**
D2. Explain your drawing	2	0	2	8	14	22	6	5	11	3	2	5	2	1	3

5.1.1. Attitudes towards mathematics

The students' responses to the questions concerning their attitude towards mathematics are coded according to the criteria found in table 10.

Table 10: Coding criteria for questions 1-5: Attitude towards mathematics.

Question / **Code**	**1**	**2**	**3**	**4**	**0**
1. What do you think of when you hear the word “mathematics”?	Emotionally negative (”it is difficult”, ”I cannot do it”, ”oh God, not that again”, ”boo”, ”oh no”	Content, specific (”multiplication”, ”fractions”)	Content, general (”numbers”, ”calculating”, ”my teacher”, ”calculations”)	Emotionally positive (”fun”, ”challenging”)	No answer / misunderstood (”nothing”)
2. How much do you like math in school?					No answer
3. Do you think you are good at math?					No answer

4. What do you like the most about math?	Don't know	Emotionally negative ("nothing")	General ("homework", "calculating")	Specific content ("division", "fractions", "math games")	No answer
5. What do you like the least about math?	Don't know	Emotionally negative ("everything")	General ("homework", "calculating")	Specific content ("division", "fractions", "math games")	No answer

In general, the majority of the 43 students (19) seem to associate mathematics with general content such as numbers or calculating, and 12 of them describe mathematics with specific content (e.g., addition, multiplication, or fractions). However, quite a few of the students (12) have emotional associations with the word "mathematics", primarily in a negative way, which is more than twice as many students with positive emotional responses (five).

When it comes to the students' enjoyment of mathematics, a large majority (31 students) find themselves in the two middle categories, most of them on the positive side of the middle. Their estimations of their own performance show a similar pattern. When asked about their most and least favorite aspects of mathematics, the students generally indicate specific content (e.g., division, fraction, math games, etc.). However, 7-8 of the students give an emotionally negative answer to these questions.

The two classes primarily differ in regard to their enjoyment of mathematics. In the X-class, only one student likes math a lot, while four students strongly dislike it. In the Y-class, the picture is the opposite. This might be connected to their feeling of being good at the subject, where the numbers are almost equal to their emotions towards mathematics.

5.1.2. Beliefs about the application of mathematics

Questions A1-A5 concern the application of mathematics and are coded according to the criteria in table 11.

Table 11: Coding criteria for questions A1-A5: Application of mathematics.

Question	Code	1	2	3	4	0
A1. Do you think it is important for you to learn math? Yes, because: / No, because:		No	Yes Unspecified justification or vague ("it is important", "you need it")	Yes Generally specified ("to get an education", "to get a job", "you use it for everything")	Yes Further specified ("I'm going to be an engineer")	No answer
A2. Do you think it is important for everybody in Denmark to learn math? Yes, because: / No, because:		No	Yes Unspecified justification or vaque ("it is important", "you need it")	Yes Generally specified ("to get an education", "to get a job", "you use it for everything")	Yes Further specified ("I'm going to be an engineer")	No answer

A3. What do you experience and think that math is used for in your daily life?	Nothing / don't know	Not specified ("everything")	Shopping + possibly something else	Several things	No answer / misunderstood
A4. What do you experience and think that math is used for in society?	Nothing / don't know	Everyday things ("Shopping")	General, society ("everything", "work", "money")	Specific, society ("to build roads", "to pay tax")	No answer / misunderstood
A5. Do you think that mathematics is more important now than 100 years ago? Why / why not?	More important 100 years ago OR equally important, no or vague justification	More important now, no or vague justification	More important 100 years ago with justification	More important now with justification	No answer / misunderstood / don't know

The students generally believe that mathematics is important for them to learn (40 out of the 43 students), yet only five of them give a specific reason why (e.g., "I'm going to be an engineer"). Among these, four of them are in the Y-class. Most of the students (27) only give a rather general or superficial explanation, indicating that mathematics is important to learn, for example, in order to get an education or a job. This picture is repeated in their opinions about whether mathematics is important for everybody to learn. Interestingly, three of the students believe that mathematics is not important to learn for everybody, even though they find it important for themselves.

In regard to the actual application of mathematics, the majority (26) of the students primarily associate mathematics with shopping. 11 students can list several things that mathematics is used for in their daily life. Only three students answer "nothing" or "I don't know", and the same number of students give a non-specified answer (e.g., "everything"). When it comes to the role of mathematics in society, the students seem to be even more unsure, with only 8 students being able to mention specific application areas of mathematics in society (e.g., "to build roads" or "to pay tax"). The rest of them are evenly distributed in giving a superficial answer (e.g., "everything") (12 students), giving an answer that indicates daily life areas instead of societal (11 students), or answering "nothing" or "I don't know" (11 students).

When asked to put the application of mathematics in a historical perspective, the students do not agree on whether it is more important now than 100 years ago. Approximately half of the students (21) believe that mathematics is more important now than 100 years ago, and 18 students believe that it was more important 100 years ago or equally important. However, while most of the students who indicate that mathematics is more important now are able to justify their answer (18 out of 21), this is only the case for one third of the students believing that it was more important before or equally important. Interestingly, and supported by the students' beliefs connected to the historical development of mathematics, there is a rather high number (six) of students who do not have an answer for this.

5.1.3. Beliefs about the historical development of mathematics

Table 12 shows the criteria for the coding of the students' answers about the historical development of mathematics.

Table 12: Coding criteria for questions B1-B5: Historical development of mathematics.

Question	Code	1	2	3	4	0
B1. How do you imagine that the math you learn at school has come into being?		Don't know or deficient response	Emotionally negative	Vague justification ("somebody just thought of it" or "to measure something")	Reflected response ("it has always been part of life", "someone felt a need for counting and calculating")	No answer / misunderstood
B2. Why do you think somebody came up with mathematics?		Don't know	Emotionally negative	Vague justification ("to make it easier", "to add 2 and 2")	Reflected response (To be able to be more precise, e.g., when building something", "because we need units to the things we do"	No answer / misunderstood
B3. When do you think mathematics came in to being? (specify year or period)		Don't know	Specific (random) year ("1773", "1900")	Period (random) ("the Iron Age")	Justified time indication ("since humans came into being")	No answer / misunderstood
B4. Did you ever learn anything in school about how mathematics came in to being? No / Yes. What was it about?		No	Yes, no specification	Yes, vague specification	Yes, precise specification	No answer / misunderstood

The students' responses to the four questions concerning the historical development of mathematics indicate that their beliefs in this area might not be developed. In questions B1 and B2 concerning the origin of mathematics, only 7 (B1) and 9 (B2) students give a reflected response to these questions, and 15 (B1) students and 21 (B2) present a vague justification. There is a rather high rate of students who do not give an actual answer to these questions (19 in question B1, and 13 in question B2), either not responding at all, answering "I don't know", or giving an emotionally negative answer. Question B1, along with question A5, has the highest occurrence (6 students) of category 0 (no answer or misunderstood) in the questionnaire. Combined with students giving an emotionally negative response (2 students from the X-class), 21 students, almost half, do not give an answer to this question. This can either indicate that these are difficult questions that the students have not previously given much consideration, or that the questions are phrased in a way that is difficult to understand.

In regard to the students' indication of the age of mathematics (question B3), it is clear that they do not have a particularly refined historical awareness. Only two students—one from each class—give a somewhat justified time indication (e.g., "since humans came into being"). Even though the phrasing of the question probably has caused a bias in the data, a surprisingly large number of the students (28 students) respond to this question with a specific—and apparently rather random—year, which suggests that they have not previously reflected on this issue. Seven students mention a period (e.g., the

Iron Age, which also seems quite random). The almost equally large number of students (27) who do not have any recollection of learning about this in school (question B4) supports this, although twice as many of them belong to the Y-class (18) as the X-class (9). 13 students answer “yes”, only three of them giving a specification and 10 giving a precise specification.

The two classes do not otherwise differ much in their answers, except when it comes to the emotionally negative answers, which is far more represented in the X-class. As the questions do not call for emotionally loaded answers, these students apparently associate mathematics with something unpleasant to a degree that initiates them to reflect this feeling in the response.

5.1.4. Beliefs about the nature of mathematics as a subject area

Coding criteria for this aspect are found in table 13.

Table 13: Coding criteria for questions C1-C4: Nature of mathematics as a subject area.

Question	Code	1	2	3	4	0
C1. What is the difference between math and other school subjects?		Don’t know / no difference / deficient response	Emotional response (”I don’t like math”, ”I like English better”)	Content (”There are more numbers in math”, ”in math you calculate”)	Working methods (”we work more with computer”, ”there is more group work”)	No answer / misunderstood
C2. What is the most important in math? Choose max. 3 things:						
C2a. To get a correct result C2b. To be able to remember C2c. To use the correct methods C2d. To come up with your own methods for solution C2e. To understand what the teacher means C2f. To be able to explain what you think C2g. To solve problems C2h. To know the multiplication tables C2i. To be able to find patterns C2j. To get good ideas C2k. Other:___		Marked				Not marked
C3. What is a mathematical problem?		Don’t know	Vague (”a problem with math”)	Backwards understanding (”when you calculate wrong”)	Sign of understanding	No answer / misunderstood
C4. What do you think a mathematician does?		Don’t know	Wrong or irrelevant response.	Vague response (”does math”, ”calculates”)	Sign of understanding (”works with mathematical problems”)	No answer / misunderstood

The students' answers regarding the nature of mathematics as a subject area indicate that they are not necessarily aware of this aspect of mathematics. In question C1, almost half the students are not able to objectively characterize the difference between mathematics and other subjects (19), and the other half (21) give rather superficial answers mostly related to the content of mathematics teaching (e.g., "there are more numbers in math" or "in math you calculate"). A large majority (41) have no or very little idea what a mathematical problem is in question C3, or what a mathematician does (35 students).

However, the students do have an idea of what they find important in mathematics (question C2), indicated by the fact that they all have an answer for this. There might be several reasons for this. For example, the multiple-choice format does not require writing or the students' own formulation of their thoughts, which makes the question quite easy to answer. Furthermore, this question is easily related to school mathematics, which very likely is a more familiar and relatable context for the students. When the options in question C2 are analyzed according to the dualistic/relativistic spectrum of mathematics-related beliefs, options a, b, c, e and h can be characterized as belonging to the dualistic end of the spectrum, while options d, f, g, i, and j belong to the more relativistic end of the spectrum (table 14):

Table 14: Options in question C2, categorized according to the spectrum of dualistic/relativistic perspective on mathematics.

Dualistic:	Relativistic:
a) To get a correct result (15)	d) To come up with your own methods for solution (22)
b) To be able to remember (8)	f) To be able to explain what you think (16)
c) To use the correct methods (12)	g) To solve problems (6)
e) To understand what the teacher means (17)	i) To be able to find patterns (1)
h) To know the multiplication tables (14)	j) To get good ideas (8)
k) Other: "To learn it" (1)	k) Other: "To keep trying" + "To try as best you can" (2)

Even though the most popular option ("To come up with your own methods for solution", chosen by 22 students) belongs to the relativistic perspective, the dualistic options have a slightly higher occurrence (67 markings) than the relativistic ones (55 markings). Only one student chooses option i) "to be able to find patterns"—an option that one could imagine would be popular amongst mathematicians. Four students have other suggestions: "to learn it", "to keep trying", "to be able to use it", and "to try as best you can". An interesting observation can be found in relation to option g): "to solve problems", which six students mark. However, all of these students answer, "I don't know" in the following question C3 concerning what a mathematical problem is. A possibility might be that some students mark the options that they believe are the ones they are expected to mark.

There are a few differences between the two classes; the most significant being "To be able to remember" (seven in class X, one in class Y).

5.1.5. Mathematical activity: drawing

Table 15: Coding criteria for questions D1-D2: Mathematical activity (drawing).

Question	Code	1	2	3	4	0
D1. Draw yourself working with mathematics that you find fun:		Emotionally negative	Traditional student role (alone, sitting by a desk, solving tasks)	Other, but in a school setting	Non-school setting	No answer / misunderstood
D2 Explain your drawing		Emotionally negative	Traditional student role (alone, sitting by a desk, solving tasks)	Other, but in a school setting	Non-school setting	No answer / misunderstood

When asked to draw themselves doing math that they find fun, 33 out of 43 students associate this with school, and 22 of them (14 from the Y-class) draw themselves in a traditional student role, sitting alone at a desk solving tasks. 11 students draw some other activity, but still in a school setting (e.g., standing by the black board or listening to the teacher). Only five students draw themselves in a non-school setting (e.g., shopping online in a foreign currency or playing computer games or chess), thus associating mathematics with activities that are not directly related to school. It thereby seems that the students to a high degree are only able to think of school activities in relation to mathematics, and that their experience with mathematics is quite restricted to a certain kind of activity. Three students do not answer, and two students' drawings reflect negative emotions (both from the X-class).

5.1.6. Overall characteristics of the students' beliefs about mathematics as a discipline

In summary, the beliefs of the students in the two classes seem to be strongly connected to school mathematics, and the students overall associate mathematics with the content taught at school. Most of them enjoy mathematics to some extent. The majority rate their own performance to be in the high end of the middle. Their beliefs about aspects of mathematics that exceed what is taught at school, generally, appear either superficial or unjustified. Although almost all of them find mathematics important to learn, they have difficulties explaining why, as they struggle with giving examples of what mathematics is used for (other than shopping). The students do not appear to have yet developed their beliefs about the historical development of mathematics, nor the nature of mathematics as a subject area. Up to 12 of the students show signs of negative emotional associations with mathematics, some of them expressing this in several of their answers, even to emotionally "neutral" questions. Also, here lies the most explicit difference between the two classes, as there are more students from the X-class in this group.

These results, of course, illustrate the beliefs of the students in a general way and can be used to inform the design of the intervention. However, the students' responses also form the basis of the selection of focus students that are interviewed for an elaboration of their beliefs. The following section concerns the interviews conducted at the beginning of the intervention in 2019.

5.2. Data analysis: pre-beliefs of focus students (pre-questionnaire + pre-interviews)

In the following sections, the beliefs of each focus student are analyzed individually in the form of cases. All four analyses follow the same structure, which begins with a description of the student that includes a short characteristic, a detailed justification of the choice of that particular student as a focus student, and a summary of the student's attitude towards mathematics and his or her learning behavior. Subsequently, the student's beliefs about the three forms of OJ are analyzed.

5.2.1. Tom, class X

Tom is a quite active student that often draws attention to himself in the classroom. Encouraged or not, he often expresses his opinion concerning various topics related to the teaching and to private or social issues. He is rather popular among his classmates, but his teacher expresses that he can be quite a challenge in the classroom, because he does not always follow the rules. She also points out that he lacks motivation and interest in the subject, even though he is competitive, performs well and is active in classroom discussions. In the questionnaire, Tom's answers appear to be rather ambiguous. While the total sum of his answers (47) places him in the yellow middle category, the individual answers reveal that while his attitude towards mathematics is negative and most of his answers often are missing or very superficial, other answers have a high degree of reflection. For example, his drawing depicts him shopping for sports equipment online in a foreign currency (figure 20).

Figure 20: Tom's drawing in question D1 ("Draw yourself working with mathematics that you find fun"). His explanation in D2 translates. "Me buying a ball [on] Amazon".

Furthermore, he indicates that his performance level is quite high. Because of this ambiguity, Tom was selected as a focus student, as the apparent complex nature of his relationship with mathematics might bring a unique and interesting perspective to the development of the students' beliefs.

The ambiguity continues in the interview with Tom, where he confirms the impression of a student that does not like mathematics very much but perceives himself as rather talented. Both in the questionnaire and in the interview, Tom repeatedly expresses that he finds mathematics "very boring", "very easy", and even "a waste of time". He prefers easy tasks, so that he can "get it over with and move on to something more useful".

Tom's beliefs about the application of mathematics

In the questionnaire, Tom categorizes mathematics as "important" and "good to know", because it is used for "pretty much everything", although he gives no examples of what it might be used for. In the interview however, he appears to perceive mathematics as mostly irrelevant and a waste of time:

24	Tom:	I don't feel that you need it so much. When you know how to give back change in [the supermarket] or you can use a ruler or something like that, I don't feel that you need much more than that.

Later, he exemplifies the use of mathematics in daily life and in society:

73	Tom:	I suppose you use it all the time. I mean, what size your clothes are. You just have to remember a number, but that is math. When you have learned that, it's fine.
(…)		
263	Maria:	Can you think of things that math is used for in society?
264	Tom:	Well […] If you buy milk. Then you use math.
(…)		
267	Maria:	[…] Do you have other ideas?
268	Tom:	How many people can be accommodated in a bus. Or what number the busses should have.

Tom clearly thinks of mathematics as useful and important, but it seems that his perception of useful mathematics is limited to numbers. Everything else is pointless to learn. His contradictive statements might be a sign of inconsistent and unstable beliefs. Moreover, the lack of diversity in his examples indicates that the justification of his beliefs is quite limited. The few examples given are restricted to shopping or counting, which could signal that Tom does not have experiences (or at least has not reflected upon his experiences) with the application of mathematics, and thus this points to non-evidentially held beliefs. Perhaps this is not uncommon for students his age. (Kloosterman, 2002) suggests that mathematics education in general may not provide sufficient opportunities for students to gain such experiences and/or reflections. This suggestion is supported by Tom's teacher, who expresses that she normally does not focus on relating mathematics to the world outside school.

Tom's beliefs about the historical development of mathematics

Likewise, Tom seems to have very few beliefs about the historical developments of mathematics. In the questionnaire, he skips most of these questions. He does state, though, that mathematics came into being at the same time as humans but, in the interview, he shows signs of confusion:

283	Tom:	[Mathematics] comes from that guy, from Egypt or wherever. Or, it has always been there.
(…)		
290	Maria:	How about something like… a coordinate system. Where do you think that originated?
291	Tom:	I don't really know. Maybe something with some ships.
(…)		
294	Maria:	[…] What makes you say ships?
295	Tom:	I don't know. That game... 'Battleships'.

The lack of coherence in Tom's statements, his hesitant answers, and the association to a game are all indications of uncertainty about the subject. It is unlikely that Tom has given the origin of mathematics much previous consideration, and he might not have developed any actual beliefs about this.

Tom's beliefs about the nature of mathematics as a subject area

A similar pattern appears in Tom's statements about the nature of mathematics as a subject area. In the questionnaire, his answers are mostly shallow and evasive. Only when asked to mark the three most important things in mathematics does Tom give "real" answers: He selects "d) to come up with your own solution method", "f) to be able to explain what you mean", and "g) to solve problems". Yet, he answers "I don't know" when asked what a mathematical problem is. These choices indicate a dualistic perspective on mathematics (cf. section 2.2.3), while Tom's statements in the interview suggests a more relativistic perspective with the criteria of success being memorization, speed, and correctness:

114	Tom:	[I feel that] we learn about fractions all the time […]
115	Maria:	Yes? Do you think it is easy, fractions? […] You just have to remember it, then it is quite easy.
(…)		
238	Maria:	You say that you are quite good at math. […] How can you feel that?

239 Tom: Well, I finish fast and I have done the difficult tasks.

There are thus signs that Tom's beliefs about the nature of mathematics are rather inconsistent. However, with the statements in the interview, Tom refers to mathematics as a school subject and to the success criteria within a school context. This might explain why they represent another view on mathematics than his answers in the questionnaire. If this is the case, Tom does not seem to connect school mathematics to mathematics as a discipline. This assumption is, to some degree, supported by one of his first comments in the interview when asked about how he feels about mathematics:

16 Maria: First, please tell me how you feel about mathematics.
17 Tom: Um… well, mathematics or the lessons?

With this comment, Tom appears to differentiate between the two. Yet, in the continuation of the dialogue, he still seems to be talking about school mathematics and his feelings towards this:

18 Maria: Well, mathematics.
19 Tom: I just find it boring.

His answers to questions C3 and C4 in the questionnaire do not point to an understanding of mathematics as a scientific discipline:

C3. What is a mathematical problem? IDK [I don't know]

C4. What do you think a mathematician does? Works with math

Thereby, the overall interpretation might be that Tom has not previously given much thought to these aspects of mathematics, nor has he been offered adequate opportunities for gaining experiences. Hence, Tom's beliefs about mathematics as a discipline are generally be characterized by a low level of justification and exemplification.

5.2.2. Molly, class X

In the questionnaire, the sum of Molly's answers is 57, which places her in the green category. She has an answer for all the questions and seems to like mathematics. In class, Molly is active, motivated and shows interest in the subject.

Molly's positive attitude towards mathematics is consistent in the questionnaire and the interview. She has a feeling of learning something useful in the mathematics lessons, either for a purpose in class or for later education. She estimates her performance level to be 3 out of 4, explaining in the interview that she increased her performance level a few years earlier when a substitute teacher showed her a different and more process-oriented approach to mathematics. This experience also improved her attitude towards the subject. In the interview, she furthermore relates her performance level to her ability to concentrate in class:

126 Molly: (…) when I work, I don't do something else, not paying attention. At my table, I have seen that they [her classmates] play games on their computers. […] Then I think, "Maybe you should pay attention, then you actually get better".

Molly is selected as a focus student partly because of the development she describes; from a student with a low performance and a negative attitude to a positive attitude, higher performance level and process-oriented approach to mathematics. However, Molly's association with mathematics seems to be restricted to numbers and arithmetic, as in her answer to the first question in the questionnaire:

A1. What do you think of when you hear the word "mathematics"? numbers, plus, minus, multiplication, division

Also her drawing illustrates this tendency (figure 21):

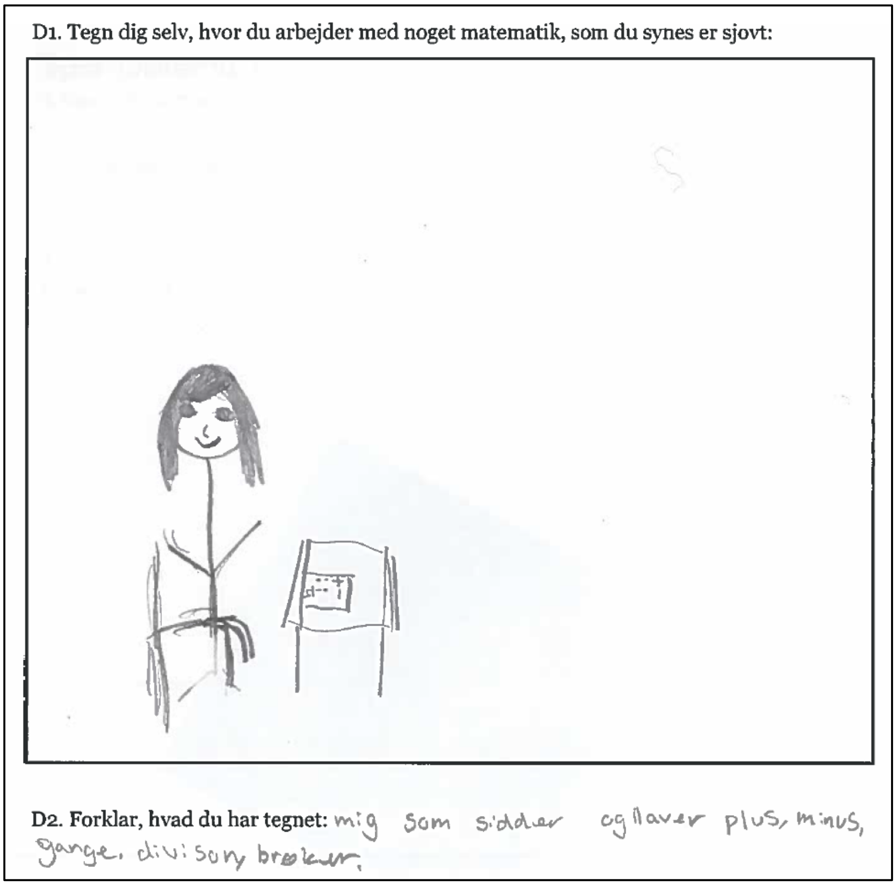


Figure 21: Molly's drawing in question D1 ("Draw yourself working with mathematics that you find fun"). Her explanation translates as: "Me sitting and doing addition, subtraction, multiplication, division, fractions".

Molly's beliefs about the application of mathematics

Overall, Molly's statements regarding the application of mathematics are both exemplified and justified. Furthermore, her beliefs seem quite consistent when comparing her answers in the questionnaire and the interview. She indicates, in the questionnaire, that mathematics is important to learn, personally (question A1) for educational purposes and because "you need it as an adult", and generally (question A2) for shopping, cooking, driving, and working. In the interview, she elaborates on this with several examples, as in the following statement where she describes an experience at her mother's job:

138	Molly:	I remember when I [helped my mom at the] pharmacy. There, you need numbers to figure it out. […] Like with fractions, [the medicine] is on shelves, then she [my mother] says "remember to put it at 2.6." Then I'm thinking "[…] I just have to put it there, like fractions. […] Okay, it's a lot about mathematics."

Molly tells me that after responding to the questionnaire, she talked to her parents about the application of mathematics, which clearly made her reflect more on the subject:

144	Molly:	My dad […] says that when you are driving, you need to know how many kilometers there are and how many minutes it takes. I also thought about that. […]
147	Maria:	Everybody […] learns mathematics in school. […] How is that do you think?
148	Molly:	Well, I think it's so that everybody has a chance to learn it.
149	Maria:	Yes? What do you use it for?
150	Molly:	Well […] I think you need it for education and work. […] For example, my dad works at the university, and there is a lot with numbers. And he has to report to the tax authorities.

Later, she also mentions how people need mathematics when receiving their salary and paying rent. Molly also reflects on her own use of mathematics in her daily life:

154	Molly:	My brother and I, we have these Thursdays where we cook. […] Then we measure things and all that, and then [I think] "we actually learned this in school today", and then I explain it to him.

Even though Molly's beliefs about mathematics generally seem to be related mostly to numbers and arithmetic, her reflections on the application are to a large extent build on examples and experiences, which she refers to when justifying her statements. Hence, her beliefs appear to some extent to be evidentially held, and to have a high degree of consistency, exemplification, and justification.

Molly's beliefs about the historical development of mathematics

Molly's association of mathematics with numbers and arithmetic becomes even clearer in her statements about the historical development of mathematics. When asked about the origin of mathematics in the questionnaire, she answers: "I think that there was a man/woman who wanted to calculate", and when asked about the reason for the invention of mathematics, Molly's answer is "to calculate and learn something." In the interview, she repeats this:

168	Molly:	It might be that there was a lot about not being able to calculate. And there were things, where you couldn't calculate it when paying for something. And somebody figured out adding it.

As in the case of Tom, Molly's answers in the questionnaire and in the interview indicate that this is an aspect of mathematics that she has not given much thought to. On several occasions, she hesitates in her answers, indicating that she is constructing her beliefs while talking:

160	Molly:	I think that earlier you didn't use it [mathematics] as much, because maybe they didn't have the same as you have now.
161	Maria:	The same what?
162	Molly:	There might be new things now that … um … maybe … that's a good question.
163	Maria:	Yes. Well, I don't know the answer, so just say what you think.
164	Molly:	Maybe there has been a new development in the schools.

The last remark in this dialogue indicates that Molly primarily perceives mathematics as a school subject. Later, when asked about when she thinks mathematics came into being, she (in the questionnaire) answers "the Stone Age". She again seems uncertain:

170	Molly:	It might be the Roman Era, I don't know. I think it is the Roman Era.
171	Maria:	Yes? Why do you think that?
172	Molly:	Because when we saw that film [unknown], there was a lot about … there was this man, who just sat there. With these … these ball things …
173	Maria:	Abacus?
174	Molly:	Yes. And then he took out the abacus and wrote in the sand. And then I thought, "okay, was that when it was invented?" Because I actually thought that it was further ahead in time. […] So I was quite surprised.

In this quote, Molly seeks to justify her answer by referring to a film previously shown in class (which might be the BBC documentary "The Story of 1" (Murphy, 2005), which Molly and several other students refer to in the questionnaire). However, it is not clear how this reference gives her reason to believe that mathematics was invented in the Roman Era. Her last remarks about being surprised that mathematics was invented that early in time, contradicts her answer in the questionnaire about mathematics coming into being in the Stone Age – provided that she is aware of the chronology of the different ages.

Overall, Molly's beliefs about the historical development of mathematics appear inconsistent, and she struggles to provide examples or justification to support her statements. Like Tom, it is thus not likely that Molly has had opportunities to reflect on the matter, nor that she has been presented with evidence on which her beliefs could be built. However, she is consistent in relating mathematics to numbers and arithmetic and seems to think of mathematics as primarily a school subject.

Molly's beliefs about the nature of mathematics as a subject area

From Molly's answers in the questionnaire and the interview, her association of mathematics with a school context becomes even clearer. In the questionnaire she describes a mathematical problem (question C3) as "a problem in mathematics about calculating", and when asked about the difference between mathematics and other subjects (question C1), she answers, "you learn to add and subtract" in the questionnaire. But when asked to elaborate in the interview, she also relates mathematics to a different way of working:

180	Molly:	In math, you write down all kinds of numbers. […] Then you sit and really work forward with it. Compared to Danish, where you have to write down all the time. In math, you write at your own pace […]. I think that's better.

Apparently, Molly has another feeling of speed and progress in mathematics than, for example, in the Danish lessons, and there is a clear process-orientation in some of her statements. For example, in question C2 in the questionnaire, she marks the three options: "d) to come up with your own methods for solution", "e) to understand what the teacher means", and "f) to be able to explain what you think", as most important in mathematics. It might be a sign that Molly is in the process of changing her focus from being correct to understanding, i.e., towards the relativistic end of the spectrum, although this is not supported by any of her other statements.

Molly's statements generally reflect that her beliefs about mathematics as a discipline are formed in the classroom. Her experiences here especially form her perception of the nature of mathematics as a subject area, which therefore might be equivalent to her beliefs about school mathematics. Hence, she relates mathematical content to the content taught in school, mathematical methods to the methods used in school, and the goal of mathematics to the success criteria in school—all of which she can both exemplify and justify based on personal experience.

In summary, Molly's beliefs about mathematics as a discipline are based on her experiences in school, as well as in her daily life. They overall appear reflected, and she is able to exemplify some of her statements, especially regarding the application of mathematics. However, she seems to have a rather narrow understanding of the content of mathematics, which she associates with numbers and arithmetic calculations, and she does not seem to have developed her beliefs about the historical development of mathematics yet.

5.2.3. Erica, class Y

Erica is a student who, in the beginning of the intervention, seems to have a rather negative attitude towards mathematics. In the first classroom observations, she appears to give up easily and to have very low self-efficacy. She does not participate in classroom discussions, but she is often verbal about her negative attitude towards the subject. The questionnaire confirms this attitude, as her answers to 11 out of the 20 questions are either "I don't know" (questions 4, A3, A4, B1, B2, C1 and C3) or they have a negative wording (questions 1, 2, 3, 4, 5 and B2). She is very explicit in her dislike for the subject, for example, by marking the unhappiest emoticon in question 2 or answering "Everything", when asked what she likes the least about mathematics in question 5. The rest of her answers are superficial and do not indicate reflection. Consequently, the coding of her answers results in a total sum of 32, which is the lowest in her class. She was thus selected for an elaborating interview.

However, when I interviewed Erica, there seems to have been a remarkable change in her attitude during the relatively short time (approximately two months) since she responded to the questionnaire. Her emotional disposition, her vision of mathematics, and her perceived competence seem to be more positive. She now describes mathematics as fun, and she has apparently discovered that, in contrast to what she used to think, she is actually capable of doing mathematics. She seems to have increased her self-efficacy and now works with more confidence and perseverance:

32	Erica:	Well… it might seem troublesome of course, but you are certainly able to learn it, you just have to be willing to learn it […]. And that is something I have worked really hard to do […].

The choice of Erica as a focus student is thus based on her low score in the questionnaire combined with her remarkable change in attitude.

Erica's beliefs about the application of mathematics

In spite of the changes in Erica's attitude towards mathematics, her beliefs about the application of mathematics seem to be more or less consistent from the questionnaire to the interview. However, the character of her answers differs, as they are very short and shallow in the questionnaire and more elaborative in the interview.

In the questionnaire, Erica states that it is important to learn mathematics for a future purpose:

A1. *Do you think it is important for you to learn math?* Yes, ***because:*** I can use it in the future.

A2. *Do you think it is important for everybody in Denmark to learn math?* Yes, ***because:*** you can use it later after finishing school.

However, she answers, "I don't know" when asked what mathematics is used for in her daily life and in society (question A3 and A4).

In the interview, Erica still primarily finds mathematics useful for educational purposes:

219	Maria:	What do you think you can use mathematics for?
220	Erica:	Um … well, when I'm getting an education or when I'm going to high school […] Then I need an exam.

She mentions that mathematics is "really important" and that "everybody needs to learn it". Yet, she does not imagine herself needing mathematics in the future in life or in her dream job as a riding instructor:

229	Maria:	Do you think you will use mathematics when you finish school?
230	Erica:	Um, not for the education that I want.
231	Maria:	What would you like to do?
232	Erica:	I am going to be a riding instructor.
(…)		
235	Maria:	And you don't need mathematics for that?
236	Erica:	Not really. You need it a little bit.

Even though Erica might not imagine herself needing mathematics, she recognizes that other people use it in daily life for shopping, counting, and calculating, as well as in their jobs:

252	Maria:	Do you think your parents use mathematics?
253	Erica:	Well, my dad does. He is a chef, so he uses it a lot.
254	Maria:	Okay.
255	Erica:	And my mom also uses it. She is a doctor, so she uses it very, very much. So yes, my parents use it. And so do my stepparents. A lot.

Interestingly, Erica is in fact able to recognize the application of mathematics when the question includes a context, as when asked about her parents' jobs. In contrast, when asked about the application of mathematics in society, Erica cannot think of any examples.

When judging from the collected data, it seems that Erica's beliefs about the first form of overview and judgment are based on very little experience and reflection. She contradicts herself in relation to the importance of learning mathematics and, apart from her parents' jobs, she is unable to present more than a few simple examples of the use of mathematics in daily life.

Erica's beliefs about the historical development of mathematics

Erica's beliefs about the historical development of mathematics are practically non-existent. In the questionnaire, she answers "I don't know" to nearly all the questions on this matter, except question B3, where she answers that mathematics came in to being in the Iron Age. When asked the same question in the interview, her answer shows that this is not something she is sure of: "the Middle Ages, I think it's called. The Vikings, I think, I don't know. Around that time." However, she later remembers that they talked about the Egyptians in school: "They at least started some mathematics. And it has just spread to the whole world." It thus appears that Erica does not have much knowledge or experience to draw on in this matter, and hence her beliefs about the historical development are yet to be developed.

Erica's beliefs about the nature of mathematics as a subject area

There are indications that Erica's beliefs concerning the nature of mathematics as a subject area are similarly in the process of developing. However, they seem to have already started this development during the time between the questionnaire and the interview. In the questionnaire, Erica answers, "I don't know" to question C1 (the difference between mathematics and other subjects), and C3 (a mathematical problem), and to question C4 about what a mathematician does, she simply writes "does mathematics". In the interview however, she states that "there is a big difference" compared to other subjects, describing mathematics as more difficult, not only for herself, but for all students in her class. Furthermore, Erica finds that mathematics is learned in a different way, indicating a pattern where the teaching is organized based on topics and repetition:

301	Erica:	[…] [our teacher] tries to do something different, so that we don't get bored, but mostly we just do the same so that we learn it. And then we do that for a period and then we start something new. And then we try that. […]

Erica sees a good student as one who can always give the correct answers, and her drawing indicates that she associates mathematics with arithmetic (figure 22).

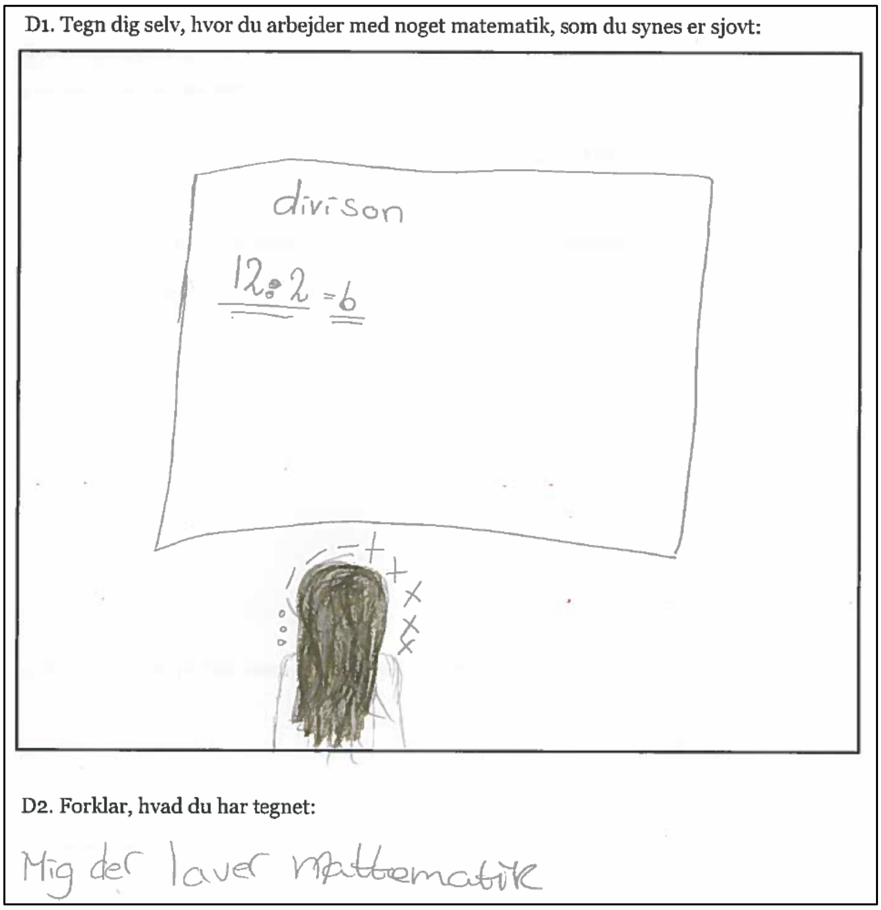


Figure 22: Erica's drawing in question D1 ("Draw yourself working with mathematics that you find fun"). Her explanation for D2 translated is: "Me doing mathematics".

In question C2, she further finds that "c) to use the correct methods", "e) to understand what the teacher means", and "j) to get good ideas" are the most important in mathematics. Where the two former options indicate a quite dualistic perspective on mathematics, the latter can be perceived as more relativistic. However, when asked to elaborate on her choice of these three options, they all appear to belong to the dualistic end of the belief spectrum:

307	Erica:	I think it is important to use the correct methods, because then you do not learn it in one way that might not be right. And then you try again and again and again, and you just get the same answer, so it's important to get a good method.
308	Maria:	Yes?
309	Erica:	And … to understand what the teacher means … I guess it's that you understand it before you can start your task. Because, I think that is really important—that I understand it before, so I don't do all the tasks and discover that it's all wrong. That is not cool.
310	Maria:	No.

311 Erica: To get good ideas …

312 Maria: Yes. What do you think here?

313 Erica: Well, I think it's important to get good ideas in math to learn something new and to try something else instead of just doing the same.

314 Maria: Yes? What do you need good ideas for?

315 Erica: Um, well, [our teacher], she has used good ideas, for example, to do 'number of the day' and different fun games.

Thus, the last option turns out to be related to the teacher's planning skills and not to students' mathematical activity or mathematical problem solving. Still, there are small signs that Erica's beliefs might be changing, as in the following dialogue about working in groups:

99 Maria: What about working with others? Do you like that?

100 Erica: Yes, in math […]. In Danish, I like to work alone, because I have my own answers, but in math, you can give different answers.

101 Maria: Yes?

102 Erica: So that you won't keep doing it the wrong way, or keep doing it in the way that calculates it wrong. So I think it's nice to work in groups in math, but not in other subjects, because then I have my own answers and opinions.

Although Erica obviously does not think of mathematics as a subject area that includes opinions or multiple solutions, she sees an advantage in discussing with others. Still, the last sentence in line 102 clearly implies that Erica does not perceive mathematics as a subject that includes personal answers or opinions.

As in the case of Molly, Erica's beliefs about the nature of mathematics are primarily based on experiences and norms in the classroom, and she does not seem to be aware of the characteristics of mathematics as a scientific discipline. In summary, she generally has very few examples connected to mathematics as a discipline and her beliefs appear somewhat inconsistent and unjustified.

5.2.4. Adam, class Y

With a coding sum of 46.5 (the half point is from question 2, where Adam marked his enjoyment of mathematics between the two middle options), Adam falls in to the yellow (middle) category. In the questionnaire, his answers are generally as short as possible and rather shallow, as if he attempted to complete the questionnaire with as little effort as possible. Examples of this can be found in question A3 and A4 concerning what mathematics is used for, where he simply answers "Everything". He has one expression of negative emotions in the questionnaire:

1. *What do you think of when you hear the word "mathematics"?* Oh God, not that again

When he is asked to elaborate on this in the interview, he says that it was an expression of his mood that particular day, but that he normally feels okay about mathematics. In the classroom, Adam is a student that does not appear to be particularly interested or engaged in the subject of mathematics, as he does not participate in discussions and seems quite indifferent about the tasks. He seems to work with the tasks to the required level, still leaving the impression that he wants to put in as little effort as he can. This impression is upheld by Adam's statements in the interview, where he confirms that he likes finishing as fast as possible and chooses tasks with the least challenge. He explains that he does not raise his hand, even if he knows the answer to a question. Adam feels that his performance level has decreased during the latest years, but lately he has started working more focused, as if he decided to take it more seriously:

47	Maria:	If I ask you how you work in math – can you tell me about that?
48	Adam:	Yes. Well … I started working a lot; lately I started working a lot.
49	Maria:	Yes?
50	Adam:	And try to do like really a lot. But a while ago I didn't do much, but I started that lately.
51	Maria:	Okay? What happened, since you suddenly started working more?
52	Adam:	I don't know. It just started.

Overall, his answers in the interview are very short, hesitant, and what one might call passive. He mostly answers "I don't know" or "I don't remember", even when it comes to preferred learning conditions or favorite activities, and he cannot think of anything that he finds fun to do.

As a focus student, Adam is thus a student that appears rather indifferent and unreflected when it comes to mathematics. On the other hand, he does not seem to associate mathematics with strong emotions, nor are there any indications of any learning difficulties. It is interesting to see if the changed focus in the intervention may increase his motivation and engagement in the subject.

Adam's beliefs about the application of mathematics

As indicated above, Adam believes that mathematics is used for "everything", both in his daily life and in society. When I ask him to exemplify this in the interview, he remembers how he, at one point, used mathematics:

116	Adam:	Once, I used a fair amount of mathematics. There was a time that I was gaming a lot, and then I both made some video edits, where I used some [mathematics], and I also programmed some games; and there I have used …. Well, one of my

games—it’s a very small game, but it took me a long time, where I just had to calculate and put it together.

Apart from this, he struggles to come up with examples of the application of mathematics:

119 Maria: What else do you think you can use mathematics for?
120 Adam: I don’t really know. I mean, I think that there is a lot of mathematics everywhere, so …
121 Maria: Yes? What might that be?
122 Adam: Well, just to go shopping or something like that, and … yes …
123 Maria: Do any of your parents used math in their everyday life?
124 Adam: Um … yes, probably. I think so. I don’t really notice it.
125 Maria: Do any of them use it for work?
126: Adam: Well, yes, maybe a little. But … I actually don’t know.

Hence, Adam seems convinced that mathematics is essential, but cannot explain why. In the questionnaire, he even states that society would “collapse” if people did not learn mathematics. When I ask him to elaborate, his explanation mostly relates to money, and it comes off rather diffuse and incoherent:

136 Adam: Well, because… if you could not calculate things, then… then you would… first of all, money wouldn’t be able to circulate, if people couldn’t calculate. And if you did not have money, then food and goods, things would have to be free, if you didn’t have money. And if it was free, some might take too much, maybe. But still, if you did not have mathematics, you couldn’t […] figure out, how much people should have, if it was all free.

For Adam personally, he finds mathematics important for his future job options. He would like to work on a ship, and he has an idea that mathematics may be used for that, stating, "I think there are a lot of machines and things like that.”

Even though Adam on one hand thinks that mathematics is very important and that it is used for everything, and on the other hand finds it difficult to specify this, he is able to reflect on the matter during our conversation, as his example of designing computer games showed. Later in the interview, there is another example of this, although a hesitant one:

170 Adam: When you think about it, there is also a lot of mathematics in the other school subjects.
171 Maria: There is?

172 Adam: Yes … well … in craft and design you have to measure and stuff like that.

(…)

175 Maria: So you use it sometimes in the other subjects?

176 Adam: Yes, I think so. But I don't know. I don't really know, actually.

This example is quite illustrative of how Adam does not feel sure in his answers and it gives a strong impression that he has not previously given the application of mathematics much thought or attention.

Adam's beliefs about the historical development of mathematics

Also, when it comes to the historical development of mathematics, Adam appears unsure and hesitant. However, his uncertainty may lead to reflection, as in the questionnaire where he gave one of the most reflected answers to question A4. He sees the matter from more than one perspective:

A4. *What do you experience and think that math is used for in society?* Yes, because there is more math today than ever before. No, because we have so many tools.

Unfortunately, he does not remember giving this answer when asked about it in the interview, and he is not able to elaborate on his thoughts. He does however have a few ideas about the origin of mathematics, both in the questionnaire:

B3. When do you think mathematics came in to being? (specify year or period) In the old Egypt. The Egyptians were some of the first to make numbers.

—and in the interview, where his statements are partly consistent with the questionnaire:

154 Adam: I heard that [mathematics] came from Egypt. Or at least … they already had it then. But I don't know where it comes from. I just heard it.

(…)

159 Maria: What if you should guess? If you should try to imagine how it came into being?

160 Adam: Well, mathematics has always been there in some way, I think.

161 Maria: Yes? Has anybody invented it, or is it something that has been discovered?

162 Adam: I think that it is something that has been discovered, more than invented.

163 Maria: Yes, okay. Something like a coordinate system, where do you think that stems from?

164 Adam: Um … many, many years ago when you had to coordinate, and where you should go and … maps, maybe? I don't know.

In spite of Adam's uncertainty and hesitation, he is actually able to base a few of his statements on examples that he remembers to have heard of, and he is able to reflect on the matter that we discuss.

Adam's beliefs about the nature of mathematics as a subject area

As in the other categories, Adam's answers concerning the nature of mathematics are very hesitant and the sentence "I don't know" appears quite often. Thereby, his beliefs about this aspect seem to be undeveloped. His answers in both the questionnaire and the interview indicate that Adam mainly associates mathematics with numbers and calculations, for example, in question C1, where he answers "a lot of numbers" when asked to describe the difference between mathematics and other subjects, or in his drawing, in which Adam is doing a calculation on a whiteboard (figure 23):

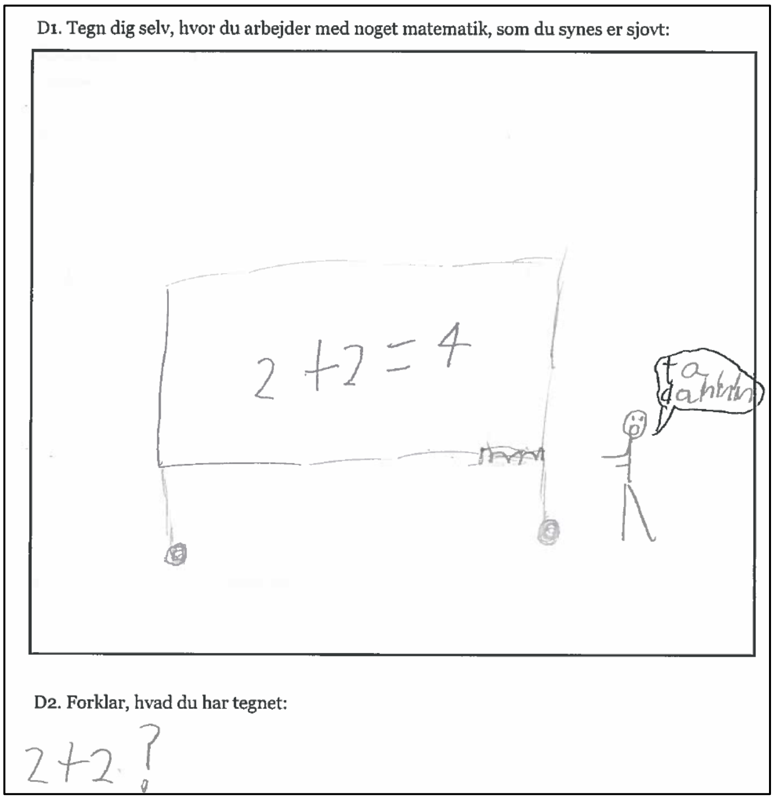


Figure 23: Adam's drawing in question D1 ("Draw yourself working with mathematics that you find fun"). The text in the speech bubble reads "Tadahhh".

He seems to find the success criteria related to correct answers and correct methods:

105	Maria:	[...] How can you see or know that someone [is good at math]?
106	Adam:	Yes, that ... well ... It's probably those, who raise their hand, and, like, almost always have the correct answer, I think. Or, I actually don't know. I don't really think about it.

In question C2, Adam lists "a) to get a correct result", "c) to use the correct methods", and "f) to be able to explain what you think" as most important in mathematics. Thus, most of his statements point to a somewhat dualistic perspective of mathematics. Others, however, are of a more relativistic character, as with his answer to question C4 in the interview, indicating a belief of mathematics as dynamic:

C4. What do you think a mathematician does? try to find new methods.

He also suggests that the way to become better at mathematics is to “try, and try hard. And keep trying”, which points to a perception of mathematical skills as something you can achieve instead of something you have. This is supported by his previously mentioned decision to start working harder.

Still, most questions posed in both the questionnaire and the interview are not actually answered with more than “I don’t know” or “I don’t remember”. Again, this could indicate that Adam has not previously reflected on the nature of mathematics. Yet, it could also be a sign of shyness or insecurity in the sense that Adam might not like making statements that he is not sure of, or commenting on issues that he does not feel he has enough knowledge about. This last theory is supported by the circumstance that he is, in fact, able to reflect on and discuss the different issues when I push him a little.

Overall, the data suggest that Adam’s beliefs about mathematics as a discipline are not yet developed, and they lack exemplification and justification. However, he shows clear signs of reflection when given time and opportunity.

Chapter 6: Subdiscussion

The purpose of research question 1 was to investigate what characterizes middle school students' beliefs about mathematics as a discipline. I have approached the investigation of this from two different perspectives:

1) A questionnaire among 43 middle school students, which provides information about the overall character of the students' beliefs in the two classes in which the intervention took place. Thereby, this part of the study not only helps to form an understanding of the students' beliefs, it also gives information to the design of the intervention. Furthermore, the findings can be compared to the results of the literature review in section 2.2.

2) An elaborative case study of four students in the participating classes; a boy and a girl from each class, representing a variety of beliefs and attitudes towards mathematics. This contributes to the research question with a deeper and more thorough understanding of the character of the students' beliefs and provides a possibility of investigating the nature of these beliefs in terms of level of consistency, exemplification, and justification as well as to which degree they are based on evidence. The main points from each case are listed in table 16.

Table 16: Main findings in the four cases of focus students, pre-beliefs.

	OJ1	**OJ2**	**OJ3**	**Overall**
Tom	Math is important and used for everything. Learning math for education and work. Does not see the relevance of learning advanced math. Low consistency. Few examples. No justification. Non-evidentially held.	Low consistency. Hesitant answers. Few examples. No justification. Non-evidentially held.	Math is mostly numbers Success criteria: speed, memorization, and correctness. Low consistency. Few examples. Low justification. Non-evidentially held.	Negative emotional associations. Relates mathematics to numbers. Beliefs about math. as a disc. not yet developed.
Molly	High consistency. Several examples. Justification. Beliefs based on evidence. Evidentially held.	Low consistency. Few examples. Low justification. Non-evidentially held.	Math is associated with numbers and calculations. Sees math as a school subject. Success criteria: correctness and speed. Evidentially held (experiences from school).	Mathematics primarily seen in a school context and based on experiences in school.
Erica	Math is important for education (exams). No feeling of future personal relevance. Somewhat consistent.	Low consistency. No examples. Low justification. Beliefs not yet developed.	Math is difficult. Success criteria: correctness. Signs of change in beliefs from schema-	Beliefs are primarily based on experiences in school.

	Few examples (none related to society). Low justification. Non-evidentially held.	Non-evidentially held.	orientation to process-orientation. Non-evidentially held.	Beliefs about math. as a disc. are yet to be developed.
Adam	Math is important and used for everything. Few examples (shopping and money), one example for personal experience. High consistency. Some justification. Non-evidentially held.	Math has always existed. High consistency. Few examples. Low justification. Very few answers (beliefs not yet developed). Non-evidentially held.	Missing answers (beliefs not yet developed). Success criteria: correct results and methods. Non-evidentially held.	Hesitant or missing answers. Beliefs about math. as a disc. are yet to be developed. Signs of reflection.

6.1. Findings and implications related to research question 1

In summary, the findings from these two data types show that the students find mathematics important to learn, both for themselves personally and for people in general. However, it seems difficult for them to express why, and they primarily see the purpose of learning mathematics as related to education or career. Furthermore, the students struggle with giving examples of the use of mathematics that exceeds shopping.

Another important, though not surprising, finding is that the students' beliefs are largely connected to school mathematics, as for example seen in the cases of Molly and Erica. When it comes to aspects of mathematics that are not directly connected to school mathematics, the students generally do not seem to have had the chance to reflect on these and develop beliefs about them. For example, as was the case with Adam, they often answer, "I don't know", or their answers are hesitant, superficial and, apparently based on guessing. It is within these aspects of mathematics that the levels of consistency, exemplification, and justification are lowest in the students' statements. Especially the students' beliefs about the nature of mathematics as a subject area seem to be based on experiences in the classroom. In that way, the students' beliefs about the nature of mathematics as a subject area becomes similar to their beliefs about the school subject with its norms and success criteria.

There may be several reasons for the strong influence of the school subject. As mentioned in the case of Tom, middle school students primarily meet mathematics in the classroom and may not have had opportunities to gain experiences with mathematics in a non-school setting, as also pointed out by Lester (2002). This reason is supported by the overall results of existing research presented in section 2.2.3, finding that students largely associate mathematics with numbers and calculations, and that their beliefs are based on their experiences in the classroom. Thereby, the findings are more or less expected, and they suggest that the mathematics education in these two middle school classes might not have presented aspects of mathematics as a discipline to an extent that has made a detectable impact on the students. Furthermore, it does not seem to have provided the necessary experiences for the students to have developed beliefs about these aspects. In a larger perspective—and considering that these aspects are encouraged in the curriculum, at least on an overall level in the purpose of the subject— one might say that the difference between mathematics as a school subject and as a discipline is alarming, and that an aim for mathematics education should be to decrease this difference.

Another reason that the students largely relate to the school subject when asked about mathematics might be connected to the methodology, particularly in terms of influencing the students' beliefs and contributing to their development through the interaction in the interview (cf. section 3.1). The questions posed in both the questionnaire and the interviews mention the school subject quite a lot (e.g., "What is the difference between mathematics and other subjects?", "If you imagine a student that is really good at math, how can you tell that he or she is good?", or "How much time would you say it is fair to spend on a mathematics task?"). Following the argumentation in the previous paragraph, the question is whether these two dimensions can be separated at this age, since school is the students' primary source for experiences with mathematics. It might be difficult to ask relatable questions that do not include mathematics as a school subject. Nevertheless, the case of Erica points to the importance of *activation* when assessing beliefs (Törner, 2002, cf. section 2.1.5), perhaps particularly students' beliefs. When directly asked about the jobs of her parents, Erica is suddenly able to relate mathematics to the world outside school. It can be difficult for an individual to describe or explain their beliefs if they are not related to a context or associated with a topic. As described in chapter 2, beliefs are structured in systems where they relate to each other (Green, 1971). Moreover, contradictory beliefs can even exist as long as they are not confronted with each other. It thus means that in order for a person to access and describe his or her beliefs, it can be necessary to "open up" a belief system by relating to an object within a belief system. When investigating a person's beliefs through an interview, a dilemma thus emerges, as it on one hand might be necessary to provide a context in order to make the informant activate his or her beliefs. On the other hand, a given context might lead the informant's thoughts in a specific direction and thereby hinder a free association. In the interviews with the focus students, little non-school context is provided, as the aim is to uncover their immediate associating and understand their association with mathematics. As the example with Erica shows, this choice affects the outcome of the interviews, as it may restrict the students' access to their beliefs. On the other hand, the data collection takes place in a school context, thereby signaling that the topic is related to school. Moreover, the questions unavoidably contain a higher degree of in-school context, as argued above. Hence, the questions inadvertently activate beliefs about mathematics as a school subject instead of as a discipline, which contributes to an indication of a large gap between these two aspects. This supports the abovementioned hypothesis that the teaching does not include mathematics as a discipline in the school subject; at least not to an extent which has provided the students with substantial experiences. Ideally, aspects of mathematics as a discipline should be incorporated into the school subject, so that the two dimensions of beliefs overlap.

Certain implications may be connected to the data collection, as they may entail possible sources of error. One lies within the process of transcription. As I did not transcribe every interview myself (due to the comprehensive amount of audio material), I may not always be aware of non-verbal messages in the dialogue that could play an important part in the interpretation of the students' beliefs. For example, it is difficult to derive irony from a written transcript. In cases of doubt, I have generally listened to the audio recordings, but there may, of course, be cases of misinterpretation. Another source of error may be found in the students' reception of questions in the questionnaire as well as the interviews. They may have misunderstood questions, or there may have been words, which they do not know the meaning of. For example, during the pre-interviews, I noticed that several of the interviewed students were unsure of the word 'society'. In the interview, I had the possibility of explaining the meaning of it, but the pre-questionnaire also included this word. Thereby some students actually answered questions of which they did not necessarily understand the meaning.

In retrospect, the phrasing in some of the questions, especially in the questionnaire, might call for reconsideration. For example, question B2 ("Why do you think somebody came up with mathematics?") implies that mathematics is invented and not discovered, which is something that the selected students are asked to consider in the subsequent interview. However, to preserve the possibility for comparison, the phrasing remains in the post-questionnaire. Still, other questions have been rephrased after careful consideration (cf. section 4.2.3)

Overall, this form of data collection has certain limitations, as the role of school mathematics is an example of. Beliefs are complex, psychological components in even more complex learning processes with both cognitive and emotional aspects. As described in chapter 3, accessing students' beliefs is thus a very complex matter, which can only happen to a certain degree and with a high level of uncertainty, as the students' statements are only mere reflections of a small part of their belief system. The students' beliefs may even be emerging, constructed, or modified in the process of investigating them, and they thus might still be subtle, volatile and difficult to articulate. Moreover, the statements must be interpreted to assess the beliefs that lie behind them, or that emerge in the context. Hence, the findings must primarily be perceived as indications and tendencies. These indications do, nevertheless, match the results of previous research on students' beliefs about mathematics as a discipline. As seen in the literature review in section 2.2.3, the dualistic perspective on mathematics is similarly prevalent among the students in this study, although there are some signs of relativistic thinking as well.

6.2. Consequences for design of the intervention

The findings presented here form an essential part of the basis for designing the intervention in this study. The necessity of providing the students with opportunities for gaining experiences with mathematics as a discipline thus becomes quite clear with the low number of examples connected to mathematics as a discipline in the students' answers and statements. There are strong indications that the students base their beliefs about mathematics primarily, if not solely, on their experiences in the classroom, and that these experiences do not seem to have included aspects of mathematics as a discipline. These beliefs have developed throughout their entire schooling, and some of them have possibly become quite central and robust. It is, therefore, essential to incorporate approaches in the design of the intervention that may make the students aware of these beliefs and make them reflect on the reasonability of these beliefs. Therefore, they must be offered experiences that exemplify aspects of mathematics related to the discipline (i.e., the actual application, the historical development, and the nature of mathematics). These experiences thus constitute a basis for either a formation of new beliefs, or for a cognitive conflict that may be solved through reflection and thus contribute to a modification of existing beliefs or a formation of new beliefs.

The students in the two classes undoubtedly find mathematics important, but they cannot explain why. Hence, the intervention needs to offer them examples of this. The statements from the focus students clearly indicate that they have not previously considered the historical development of mathematics. The intervention must thus make them aware of this aspect. As pointed out in the Danish competencies framework (cf. section 2.3.1), this must be done through concrete examples, if their beliefs are to have any weight (Niss & Højgaard, 2011, 2019). This type of overview and judgment should not be confused with a knowledge of "the history of mathematics" viewed as an independent topic. The focus is on the actual fact that mathematics has developed in culturally and socially determined environments and is subject to the motivations and mechanisms which are responsible for this development. On the other hand, it is obvious that if overview and judgment regarding this development is to have any weight, it must rest on concrete examples from the history of mathematics.

Furthermore, the students, to a large degree, equate the nature of mathematics as a subject area with mathematics as a school subject. A response to this might be what Niss (1980) suggests for upper secondary education: to include aspects of the subject area into the school subject (e.g., philosophical discussions as well as methods and processes used in the scientific discipline of mathematics such as modeling, reasoning, representation, etc.). Niss argues that by doing so, in combination with examples of its history and application, mathematics can be represented as a "multi-dimensional organism", and not "detached from time, space and society" (Niss, 1980, p. 56, my translation).

Chapter 7: Intervention

This chapter concerns the design of the intervention and includes a presentation of the teaching principles, on which the intervention was based, as well as descriptions of and arguments for the adjustments made in the four iterations. The last part of the chapter presents examples of applied learning activities.

7.1. Teaching principles

Several issues were considered in the design of the intervention in relation to beliefs theory and to didactical as well as practical considerations. Hence, the teaching principles were designed to meet certain criteria:

1. A focus on developing the students' overview and judgment. This means that all three aspects (OJ1+2+3) should be present and accessible for students.
2. The principles should reflect the theory connected to beliefs and changing of beliefs, making it essential to include experiences, concrete examples, and reflections.
3. Possibilities and obstacles connected to practice had to be considered (e.g., curriculum, teacher competences, students' proficiencies, class cultures, physical surroundings, and available materials).

In the preparatory phase of the intervention, I presented a draft for the teaching principles to the two teachers. This draft was based on a structure for design research presented in Bakker and van Eerde (2015), and included a first suggestion for goals and activities along with a brief introduction to DBR and central elements from the Danish competencies framework concerning overview and judgment. The intended purpose of the draft was to form the basis of a discussion with the teachers that would give them an opportunity to include their ideas, concerns, and perspectives from practice in the teaching principles.

The next two pages present a translated excerpt of this draft, which includes the parts of the hypothetical learning trajectory (HLT, cf. section 3.2.1) concerning goals and learning activities.

Goals: Where do we want to go?

Common Objectives, mathematics: subject areas and competencies.

In addition to the mathematical competencies, there are three forms of overview and judgment:

1. *The actual application of mathematics* in other subject and practice areas
2. *The historical development of mathematics*, both internally and in the light of society
3. *The character of mathematics as a subject area*

Overview and judgment consists partly of KNOWLEDGE about the three areas and partly of BELIEFS about mathematics as a discipline. The overall goal can be said to be developing students' beliefs about mathematics as a discipline.

How are beliefs developed?

- Beliefs change when they are compared and assessed according to **experiences** (Thompson, 1992).
- When beliefs are **based on evidence**, students become able to be critical and reflective.
- You strive for a coherent belief system (Op't Eynde et al. 2002). Evidence that does not fit into the system may be rejected. Changing beliefs based on experience can affect related beliefs.
- Lasting changes in beliefs require **reflection** (Parajes, 1992).

Which beliefs are the desired beliefs?

- We want the students to develop **evidentially based beliefs**, which are based on **experiences** and **concrete examples**.
- We want students to **reflect** in the development of beliefs.
- We want the students to perceive mathematics as broader than school mathematics. *Overview and judgment* relate to mathematics as a coherent subject area, which has a special character, history and social place, and which is used outside its own terrain for purposes that are not in themselves of a mathematical nature. Overview and judgment can thus contribute to **a nuanced image of mathematics as a discipline**.

OJ1. The actual application of mathematics

The goal is for students to gain insight into:

- that mathematics is a part of everyday life, society, and science.
- that mathematics plays a role in other subjects.
- what mathematics is used for, how it is used and why.
- what consequences it may have for the application of mathematics.

OJ2. The historical development of mathematics

The goal is for students to gain insight into:

- that mathematics is dependent on time and place, culture and society.
- the driving forces and mechanisms behind the development of mathematics, as well as the actors involved in this development.
- that man plays a significant role in the development of mathematics.

OJ3. The character of mathematics as a subject area

The goal is for students to gain insight into:

- the nature of the problems, ways of thinking and methods of mathematics.
- the type of results that mathematics delivers and what they are used for.
- the connection of mathematics to other subjects / disciplines
- the difference between mathematics and other disciplines

Learning activities:

The activities in the teaching must give the students the opportunity to gain **experiences** with the application of mathematics, the historical development of mathematics, and the character of mathematics as a subject area. This requires **concrete examples**. The activities in the teaching must give the students opportunities for **reflection** and for becoming aware of their own beliefs about mathematics as a discipline.

OJ1. The actual application of mathematics

- Working with mathematical modeling
- Experiences with the application of mathematics in everyday life, at home, among peers, in leisure life, and in family finances.
- Concrete examples of the application of mathematics in society.
- Dialogue and reflection on the role and application of mathematics and, to a lesser extent, consequences thereof.

OJ2. The historical development of mathematics

- Concrete examples of the historical development of mathematics.
- A historical perspective on well-chosen parts of the mathematical content.
- Dialogue and reflection on stakeholders, driving forces, and mechanisms behind the development.

OJ3. The character of mathematics as a subject area

- Explicit attention to:
    - Mathematical problem solving and problem-solving strategies.
    - Mathematical reasoning and methods
    - Mathematical symbol processing
    - The thinking of mathematics
- Dialogue and reflection on the type of results that mathematics provides.
- Dialogue and reflection on relationships to and differences from other subjects (disciplines).

In general:
In all teaching, it is proposed that exemplary activities are selected, which provide the opportunity for **experience formation** (preferably in the forms of inquiries, investigations, and experiments), **immersion** and **reflection**.

<u>Each module must include:</u>

- **concrete examples** of the application of mathematics and/or the historical development of mathematics.
- mathematical **problems and methods**.
- **dialogue** about the application, development and/or character of mathematics as a subject area.
- joint and / or individual **reflection.**

The three forms of overview and judgment are made the subject of explicit treatment, reflection and articulation when the opportunity arises. Such metadisciplinary discussions are suitable as training in being able to "rise above" the many concrete experiences one makes during the teaching, and such a general ability to be able to operate on several levels of knowledge is a prerequisite for being able to develop consciously and articulated overview and judgment as those we are dealing with here. (From the KOM report)

As described in section 3.2.1, a HLT often also includes a *hypothetical learning process*. However, during initial and continuous discussions with the teachers it became clear that a formulation of such a process was problematic. One of the reasons for this is the abstract character of the aims of the intervention, which are related to the three forms of OJ. An articulation of the individual steps towards 'insights' is extremely difficult, if not impossible. Beliefs are individual and subjective, which is also the case for their development. Each person has developed their beliefs based on personal and individual experiences, individual contexts, and through individual encounters with the world. Also, the students each enter the school context with individual filters through which they interpret the experiences provided by mathematics education. When aiming for developing or changing beliefs, it means that the experiences that may contribute to this are also individual, and the path of development can take various shapes and forms. An articulation of a hypothetical learning process would thus need to be general enough to fit all the individualities, which consequently affects its level of concreteness and usefulness. In the first phase of the intervention, we therefore postponed making decisions related to this part of the HLT. During the four iterations, we never experienced a need for it, nor an increased capacity for a meaningful and useful formulation of it, and so the HLT simply entailed the two elements of *goals* and *learning activities*.

7.2. Iterations and adjustment of principles

7.2.1. First iteration

The first iteration of the intervention lasted from September 2019 to December 2019.

When presented with my draft for the teaching principles, the teachers did not have any objections, additions, or corrections to the presented suggestion, and it thus became the foundation for the first iteration. However, the document was rather extensive, so to transform it into a working tool for the teachers, we decided to focus primarily on the four general principles listing central elements in the teaching:

<u>All modules must include:</u>

1. **Concrete examples** of the application of mathematics and/or the historical development of mathematics
2. Mathematical **problems** and **methods**
3. **Dialogue** about the application, the development and/or the nature of mathematics
4. Individual and/or shared **reflection**

Hence, the teachers could use these elements as a form of checklist in their planning of the teaching, and then use the goals and learning activities connected to each form of overview and judgment as support for planning the lessons.

In this first iteration, the overall teaching modules were already in place, as the teachers had planned this in advance of the school year. These modules were based on mathematical themes such as geometry, fractions, or algebra, and had a duration of 2-7 weeks. The intention was that the teachers and I discussed the goals and progression of each module with a special focus on the implementation of the teaching principles, and that the teachers were responsible for planning the individual lessons in detail.

During planning sessions and debriefings after observed lessons, we regularly discussed the challenges and successes related to the implementation. In combination with the classroom observations, these discussions pointed to potential adjustments of the principles. For example, the teachers clearly struggled with implementing the element of reflection, and the dialogues in the classrooms rarely became dynamic, but were instead more in the nature of the teacher asking questions, with single students answering. The students were clearly use to a more traditional teaching approach and sometimes appeared unsure or uncomfortable in more inquiry-based activities or when asked to reflect. The teachers also expressed that they were unsure of how to include the historical dimension of mathematics. Furthermore, they requested a fixed structure of the lessons that might support them in the implementation of the principles.

At the end of the first iteration, we thus evaluated the principles, which led to some additions for the second iteration that emphasized the importance of reflection and included structural considerations.

7.2.2. Second iteration

The four basic teaching principles remained (1–4), with the additions marked in italics below (5–7):

<u>All modules must include:</u>

1. **Concrete examples** of the application of mathematics and/or the historical development of mathematics
2. Mathematical **problems** and **methods**
3. **Dialogue** about the application, the development and/or the nature of mathematics
4. Individual and/or shared **reflection**
5. ***Reflection*** *as bearing didactical principle*
6. *High priority of* ***staging*** *and* ***wrap-up*** *in lessons (explained below)*
7. *Fixed* ***structure*** *of lessons, including daily learning* ***log***

Articulating reflection as a bearing didactical principle was intended to remind the teacher of its importance and highlight that this element, to some extent, should act as a guideline for planning the activities, as well as that including elements of reflection in the teaching should be highly prioritized. The fixed structure (principle 7) was to support this, as it included elements of reflection. Apart from one weekly lesson reserved for training mathematical skills, the remaining four lessons (two double-lessons of 90-minute duration) were to follow this structure:

I	**Math-word of the day**[7]
II	**Staging:** Short introduction to the challenge/activity of the day
III	**Challenge of the day:** The students work (preferably in small groups) with short teacher-controlled stops along the way, where the students can exchange ideas and inspiration.
IV	**Wrap-up:** The activity is concluded with a classroom discussion and/or reflections (shared or individual) concerning the students' work with the challenge, their processes, strategies, results, etc. Here, the links to the three forms of overview and judgment can be articulated and addressed. This element must be prioritized and must be implemented even if the students have not completed their work. It is essential that there is adequate time to wrap up, so that the mathematics may be highlighted and there is room for the students' reflections.
V	**Learning log:** The students note what they take away from today's lesson and how they will assess their own work effort.

Along with the additional principles, we articulated that the teachers' preparation largely should consist of planning the organization of staging and wrapping up, as well as good questions for working with the challenge of the day. Planning the content of the challenges was intended to be a collaborative matter between the teachers, with the possibility of consulting me, for example in connection to addressing the historical development of mathematics.

The second iteration was supposed to proceed from January to June 2020. Unfortunately, it was interrupted in March 2020, due to the pandemic closing the schools and changing the teaching to an online format. As this change was very overwhelming and challenging for teachers as well as for students, the intervention and the data collection was paused until August 2020. The regular contact between the teachers and me temporarily stopped, and was not resumed until June 2020, when I had individual and collective meetings with them concerning an evaluation of the intervention so far and their considerations for its continuation.

While the teacher in the Y-class attempted to follow the principles during the period of online teaching, the X-class teacher focused on repetition of previously learned material. During the evaluation of the iteration, both teachers stated that before the period of lockdown, the students generally seemed to have received the intervention positively, and that most of the students might even have benefitted from it. They particularly highlighted the structure of the lessons as very successful. The Y-class teacher had experienced that the students had become more curious, for example about the origin of mathematical

[7] Every lesson started with some kind of discussion, reflection, or activity concerning a mathematical term or concept that would be used during the lesson. For example, it might be a discussion of the meaning of a term (e.g., "chance", or "prime numbers"), a recapture of previous knowledge about a topic, a mathematical puzzle, or reflections about a number pattern.

concepts or the reasons for working with a certain activity. The X-class teacher noted that while some students struggled when asked to take an unfamiliar approach to mathematics, others welcomed it and seemed to flourish, particularly among students of average performance level.

Both teachers emphasized the benefits of the collective planning and requested that this be even more structured in the next iteration. They also expressed that the structure of the lessons worked well. However, they both felt that the daily learning log quickly had become routine for the students and thus did not generate reflection. Overall, they still struggled with implementing the elements of reflection and dialogue. This was in line with my considerations, as the classroom observations before the lockdown indicated that these elements were not adequately implemented, and I thus experienced a need for clarifying and concretizing them. Furthermore, both teachers expressed difficulties with including historical elements in the teaching. The Y-class teacher suggested that the reason for this might be that it required some kind of prior knowledge about the history of mathematics, which they did not necessarily possess, or know where to find information.

When adjusting the principles in August 2020, we therefore decided to keep the seven principles, making only one modification concerning the learning log. Instead of the students keeping a log after every lesson, we decided that they should note their expectation and existing knowledge about a mathematical theme in the beginning of a module, and then note their outcome and learning experiences at the end of it. Alternative formats for the log, such as audio or video recordings were also suggested.

In addition to the seven principles, we decided to include two initiatives concerning the planning of the teaching. This choice was primarily intended to accommodate the need felt by both the teachers and me concerning an increased collaboration on a more detailed level than what we previously had done. The first initiative was to specify and document the intended implementation of each of the principles prior to every module, in relation to the collaborative planning session. The second was an increase of my participation in the planning of the lessons, as we decided that I should be involved in at least one activity per module. The intention with this initiative was partly to ensure that the three forms of OJ were included in every module with the possibility of providing the teachers with ideas and knowledge, for example, about the history of mathematics, and partly to support and qualify the discussions about how the elements of reflection and dialogue might be implemented. Instead of leaving all detailed planning of the individual lessons to the teachers, I would thus be able to contribute to the planning with concrete ideas for activities and approaches.

7.2.3. Third iteration

After the adjustments and additions, the teaching principles in the third iteration were thus (changes in italics):

All modules must include:

1. **Concrete examples** of the application of mathematics and/or the historical development of mathematics
2. Mathematical **problems** and **methods**
3. **Dialogue** about the application, the development and/or the nature of mathematics
4. Individual and/or shared **reflection**
5. **Reflection** as bearing didactical principle
6. High priority of **staging** and **wrap-up** in lessons
7. Fixed **structure** of lessons, including learning **log** *at the beginning and end of each module.*

Documentation of the implementation of principles for every theme.
Researcher's participation in min. one activity per module

This iteration proceeded from August to December 2020. During this period, the teachers appeared increasingly comfortable and confident in the implementation of the principles. The collaborative planning took place approximately once a month, following the intended structure of discussing the overall content and purpose of the modules and planning the details of at least one activity. This initiative clearly ensured that the principles were implemented to a higher degree than in the previous iterations, perhaps most noticeable in relation to the historical dimension of mathematics.

Although the realized implementation in the classroom did not always follow the intentions, the deviations often led to meaningful discussions and modifications of our strategy. For example, one of the lessons that I had participated in planning was intended to include several opportunities for reflection but, in the classroom, the teacher omitted some of these. In the following debriefing, we discussed how we might contribute to a facilitation of such situations and decided that in the next planning session, we would include considerations about possible student answers and suitable teacher responses or questions.

The students' reception and outcome of the intervention appeared to be very different in the two classes. In the X-class, the students' motivation level generally appeared low, and they struggled with adjusting to the more "untraditional" approach to mathematics, as for example during inquiry-based activities. Often, they requested "normal" mathematics tasks. The teacher furthermore reported on an unsafe learning environment, where many students seemed uncomfortable making mistakes, and nervous of making a fool of themselves. My observations in the classroom generally gave the same impression. In contrast, the Y-class generally seemed to welcome the intervention, and the students appeared motivated and willing to engage in alternative approaches to mathematics. In this period, the teacher found that the students' ability to express themselves mathematically had increased and that they seemed to have a pronounced mathematical understanding, also in other subjects, compared to some of her previous classes.

The challenges in the third iteration were thus mostly caused by circumstances not related specifically to the mathematics teaching. At the evaluation of the principles, we therefore decided not to change them for the fourth iteration, but instead continue to focus on a successful implementation.

7.2.4. Fourth iteration

The fourth, and final, iteration of the intervention proceeded from January to June 2021. As illustrated in figure 13 in section 3.2, the teaching as well as the planning sessions were online from mid-December 2020 until mid-April 2021. In this second period of lockdown, the teachers appeared more comfortable with the online teaching format than in the first period, and they were thus prepared to continue the intervention to the extent that this format allowed.

Because of the online format and the fact that the students could not meet physically, we needed to include additional considerations in the planning of the teaching, as for example the well-being and social needs of the students. Hence, we attempted to include social interaction in online group rooms as well as mathematical investigations of objects, concepts, or conditions. This would encourage the students to be physically active. For example, a module on probability included data collection in the local neighborhood, and a module on functions included baking pancakes and calculating various correlations. One of the aims was to minimize the amount of time spent in front of the screen listening to the teacher, and a large part of the teaching consisted of group work, for example, with mathematical projects. Consequently, the wrap-up part of the lesson structure often had to be held at a minimum.

Some of the principles were easier implemented than others in this period, as for example including concrete examples of the application of mathematics and working with mathematical problems and methods (principles 1 and 2), as well as a fixed structure with a high priority of staging (principles 6 and 7). However, as mentioned, it was more difficult to implement elaborate wrap-ups, as well as to include concrete examples of the historical development of mathematics. Both dialogue and shared reflection (principles 3, 4 and 5) suffered under the online format, as these elements are highly influenced by the environment in which they take place. An online platform may feel restrictive to the spontaneity in a discussion, whereas a classroom context offers the possibility of discussing and arguing more freely, having small conversations in pairs, sensing the atmosphere, etc. Therefore, despite attempts to support the students' reflections and the dialogue concerning the three forms of OJ, it did not always succeed.

For some students, the online teaching seemed discouraging and demotivating, as their level of participation decreased, and they became more passive and quiet. In some of my conversations with the teachers, I got the impression that the parents of the students played an important role. It seemed to be especially difficult for those students whose parents were not able to stay home in this period, and also for those who experienced many quarrels with their parents. A few students seemed to thrive in the online format. One of the teachers later suggested that for some of these students, the online format might have increased the opportunity for individual work and thus released them from the social pressure of the classroom. It thereby allowed them to participate more freely, without worrying about the reactions from their classmates.

For many students, it was therefore an ambiguous experience to return to the physical classroom in April 2021. The large contrast between the isolated, individual work and the complex, social context seemed to be quite overwhelming for some of the students, and the first few weeks were largely dedicated to ensuring a safe and comfortable transfer. Subsequently, most students appeared motivated and uplifted by the company of their classmates, and the intervention proceeded as planned.

7.3. Teachers' evaluation of the intervention

In the final evaluation, the teachers and I discussed different aspects of the intervention, including their perception of the students' outcomes, their own experiences and outcomes, and their perspectives on mathematics teaching after participating.

Both teachers stated that they had experienced a clear change in the students' approach to mathematics during the intervention. According to the teachers, the students now had a more nuanced view of mathematics in relation to its application, history, and nature. In addition, both teachers expressed that the students' ability to work independently and with initiative had improved remarkably, and that they were able to see the relationships between mathematics and other subjects, as well as relationships between mathematical content areas. The Y-class teacher reported on a large increase in the students' motivation and self-confidence, compared to the beginning of the intervention, and pointed to this as one of the most successful outcomes of the intervention: that the students now enjoyed mathematics and were often eager to engage in it without worrying about making mistakes. Based on her long experience as a teacher, she had noticed that this class had a better understanding of mathematics than other classes, and that their insights into the nature of mathematics were much more explicit and nuanced. The X-class teacher pointed out that the social difficulties in her class had been an obstacle for the implementation, as the students had not felt safe enough to engage in unfamiliar approaches. Therefore, she did not recognize the high level of motivation that the teacher in the Y-class described, and she emphasized the importance of considering the social environment in a class before engaging in an intervention like this.

Asked about their thoughts on what should have been done differently, they both mentioned the communication of the intervention's aim to the students, and that this should have been included in the lessons regularly. The X-class teacher also suggested that the students' documentation (e.g., their notes, learning log, etc.) should have been conducted in a more structured manner.

The teachers both agreed that the intervention had been valuable for them in their professional lives. The X-class teacher who, as mentioned, was rather new to teaching expressed that the collaboration with a colleague and a researcher had helped her in her planning, given her inspiration to alternative activities, and provided her with a feeling of safety in relation to trying out new approaches, as these were "approved" by someone more experienced. Moreover, she emphasized that the intervention had made her realize the importance of relating the content in the mathematics lessons to the world outside, including a historical perspective. The Y-class teacher highlighted that the intervention and the collaboration had made her more conscious of her choices and approaches, directing her attention towards the importance of relating mathematics to its application and its historical development. Furthermore, she now felt braver in her teaching in the sense that she had previously been hesitant to include abstract mathematics or mathematical content (e.g., mathematical proofs) with which she did not feel confident. Now, she perceived this as opportunities to conduct mathematical investigations with the students. Both teachers planned to continue the teaching approach from the intervention in the following school year, and the X-class teacher already started to do so in another 6th grade class. Here, she found that the students responded very well; they had improved their approach to mathematics, for example, in relation to their ability to work inquiry-based, and their mathematical dialogues.

Overall, the teachers' evaluation of the intervention was thus positive in relation to the students' outcome as well as their own.

7.4. Examples of activities

This section presents examples of the activities that were planned based on the teaching principles. These examples illustrate how the teaching principles can potentially be implemented, and they therefore primarily represent the last two iterations, as the principles as well as our experience had been refined during the first year of intervention. Furthermore, it was, as described above, in the third iteration that we initiated my participation in planning individual lessons. Some of the following activities (1 and 3) are examples of that, and describe the *intended* implementation, and not the realized implementation in the classrooms. Examples 2 and 4 were based on collaborative discussions, but were primarily planned by the teachers, and the descriptions are of the *realized* implementation. Additional examples of activities can be found as part of the case analyses in section 8.2.

7.4.1. Example 1: The distribution problem.

(Conducted in August 2020)

The topic of a 90-minute lesson is probability. Addressing the historical development of mathematics (OJ2), the lesson is based on Pascal and Fermat's approaches to solving the question of distributing stakes in an unfinished game of chance, as presented in a simplified version by Berlinghoff and Gouvea (2004). The problem involves a two-player game of flipping a coin. Each player begins by placing a stake of €10, and then tosses the coin in turn. If the coin shows heads, the player tossing the coin receives a point; if tails, the other player wins a point. The first player to reach three points, wins the game. However, the problem occurs as the game is interrupted, when the score is 2-1 in favor of the player about to toss the coin, and the distribution of the €20 stakes is to be decided.

This lesson consists of five elements: staging, investigation, presentation of solutions, reflection, and wrap-up.

1. **Staging**: Presentation of the game, the distribution problem, and Pascal and Fermat. Told as a story by the teacher.
   The purpose of this element is to engage the students by telling a story and inviting them to engage in the game. Also, the story introduces the historical people, and encourages the students to consider possible distributions of the stakes.
2. **Investigation**: Equipped with a coin, the students play the game in pairs, considering and discussing how the stakes might be distributed fairly. The pairs present their suggestions for the class. Pairs with deviating solutions are put together in groups of four to discuss their arguments and attempt to agree on a shared solution.
   The purpose is to acquaint the students with the game and make them express their immediate ideas for distribution of the stakes. When presented with other suggestions, the students may become aware that there might be different solutions and different arguments. By including mathematical as well as non-mathematical argumentation, this experience addresses OJ3, the nature of mathematics as a subject area.
3. **Presentation** of the agreed solutions: The groups of four present their solutions along with the considerations and arguments on which they are based, and the strategies used to reach them.
   By including arguments and strategies, this element is to emphasize that methods and reasoning are essential aspects of mathematical problem solving, and that they play an important role when making qualified decisions. Thereby, this element also addresses OJ3.
4. **Shared reflection**: In a teacher-led classroom discussion, the students consider relevant issues, for example, if some of the presented solutions, methods, or arguments are better or more valuable than

others, or if the students can agree on one solution. During this discussion, the teacher presents the methods used by Pascal and Fermat, and the class discusses if they resemble the methods used by the students.

This element provides the students with an opportunity to reflect on their own strategies, the mathematical ideas behind the different solutions, and the validity of mathematical arguments. Comparing their ideas to the methods used in a historical context places the problem in both a historical and mathematical perspective. The character of mathematical methods are thereby exemplified, showing the students that they are capable of engaging in problems that "real" mathematicians struggled with.

5. **Wrap-up**: The teacher describes the historical development of probability theory, which was initiated with the work of Pascal and Fermat, and followed by important theories, for example about expected outcome of the law of large numbers—theories that are now applied in many fields, such as insurance, law, medicine, etc.

   The mathematical content area of probability is thereby inserted in a context that illustrates and exemplifies the role of mathematics in the world, thus addressing the application of mathematics (OJ1).

7.4.2. Example 2: Pancakes

(Conducted in February 2021)

This activity was part of a module concerning functions, which was conducted during a period of online teaching. The teacher had asked the parents to purchase ingredients for pancakes according to a distributed recipe, or alternatively a recipe of choice. In the beginning of the lesson, the students were asked to bring their computer to the kitchen, fetch the needed ingredients and utensils, and begin making pancakes.

There were several aims of this activity. One was to work with correlations and functions in an everyday context and apply mathematics to a real and familiar situation (OJ1). Another aim was to activate the students who spent most of their school time in front of their computers. Their level of motivation was generally dropping, so a third aim was to increase this by doing something different and fun.

After baking a portion of pancakes, the students were asked to come up with possible correlations, for example, between the number of pancakes and the number of eggs used in the recipe, the diameter of the pan, the number of portions, or the price of the ingredients. Along with photos of the pancakes, these correlations were posted on a shared digital noticeboard (a *padlet*), using different representations: text, tables, graphs, or function equations.

7.4.3. Example 3: Sum of angles in a triangle.

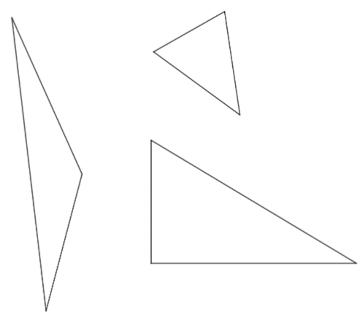
(Conducted in April 2021)

The overall goals of this activity are:

- to gain knowledge about the sum of angles in a triangle
- to experience different strategies for investigation
- to make the students aware of the character of a mathematical proof and discuss when an argument is considered convincing.

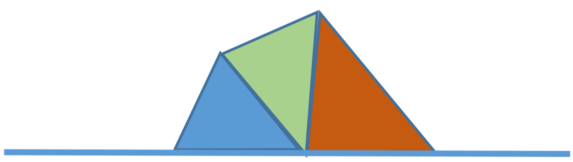
The activity thus primarily addresses OJ3, but it also includes a presentation of Euclid's proof of the sum of angles in a triangle, thereby including a concrete example of the historical development of mathematics (OJ2). The students work in groups of three or four, investigating the angle sum in a triangle using four different strategies. After each type of proof, the students discuss its validity in small groups, followed by classroom discussions about whether the argument is convincing, and if it applies to all triangles. The activity thus includes the principles concerning dialogue and reflection, and it follows the structure of staging, challenge and wrap-up.

1. Investigation, pen and paper. Measuring and adding the angles.
   The students investigate the sum of angles in a triangle by measuring and adding the angles of preprinted triangles: an obtuse-angle triangle, a pointed triangle, and a right-angle triangle.

   The groups share the results with the class and discuss: Are your results similar? Why/Why not? Can we be sure that this applies to all triangles?

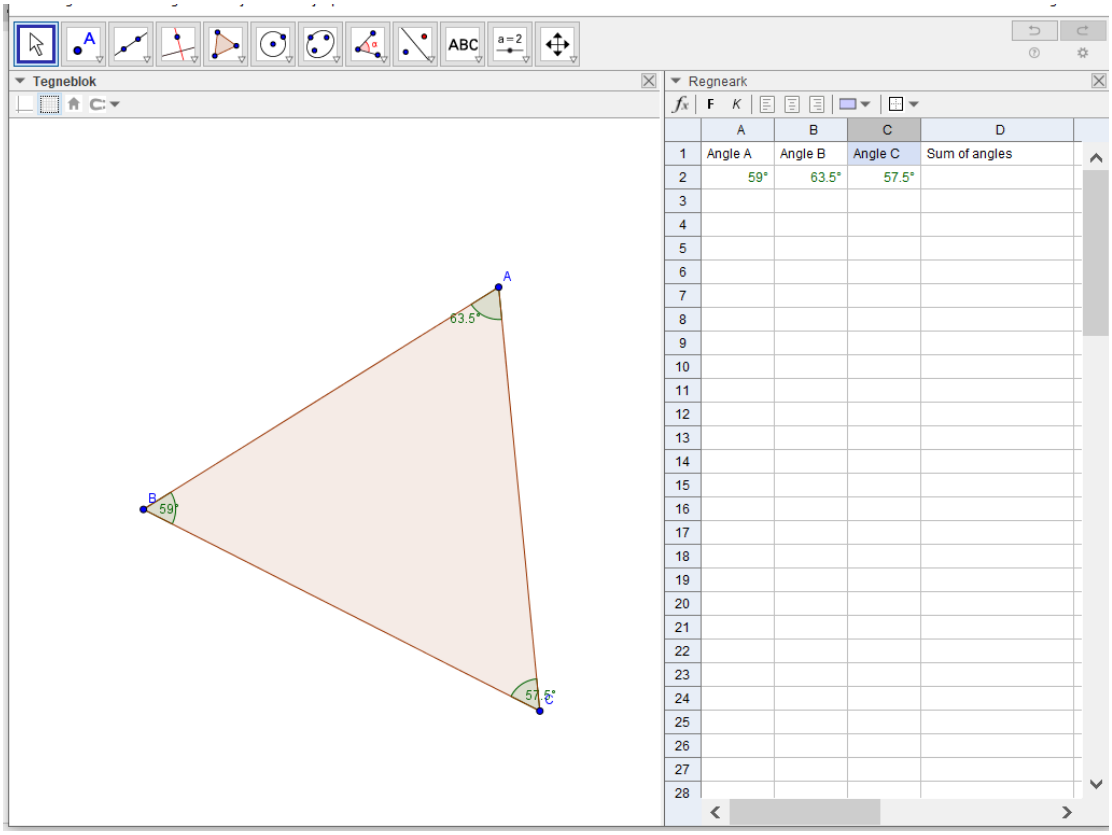
2. Proof by cutting corners:
   What does 180 degrees look like? The students construct a triangle out of cardboard. The corners are cut off and put together along a straight line.

   Discussion in groups, then in class: Can this always be done? Are we certain now? Is this a proof? Can we be sure that this applies to all triangles? What is the difference between convincing and proving?

3. Investigation, dynamic geometry system (GeoGebra):

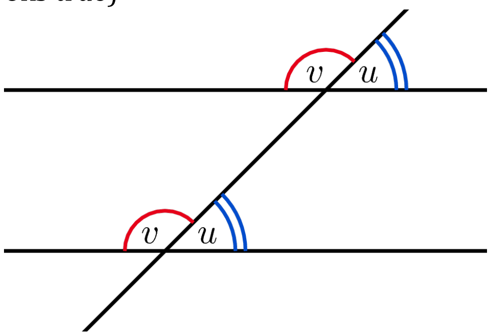
   The students construct a triangle in Geogebra and creates a sheet that sums the angles. They then use the dragging tool to change the size and shape of the triangle, thereby investigating the sum of angles in an indefinite number of cases.



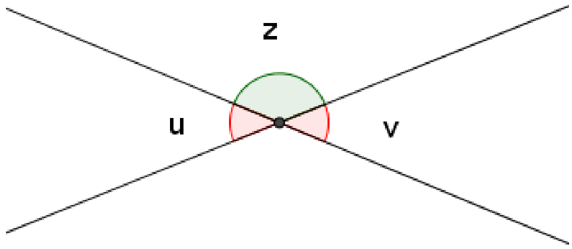
   Discussion in groups, then in class: Why does the sum seem to always be 180 degrees? Can we now be sure that this counts for ALL triangles? Why/why not? Is this convincing? Is this a proof?

4. Studying the proof found in of Euclid's *Elements,* Book I, Proposition 32 (e.g. in Heath, 1956).

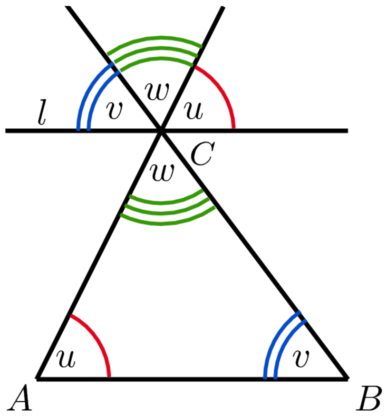
   a. First a presentation of proposition 29, Book I: If two straight lines are parallel, then a straight line that meets them makes the alternate angles equal (maybe check in Geogebra if it looks true)



b. Then a presentation of Euclid's proposition 15, Book I: *Two straight lines make equal opposite angles.*



c. Then a proof of the sum of angles by combining proposition 29 and 15.



Group/classroom discussion: Can we now be sure? Why/why not? Is this convincing? Is this a proof? Can we be sure that this applies to all triangles?

The activity is wrapped up with a final classroom discussion: What characterizes a proof, and why are proofs important in mathematics? What can they be used for?

7.4.4. Example 4: Planning a party

(Conducted in May 2021)

This project is a continuation of previous work in the classes with mathematical argumentation and using mathematics as a tool for making qualified decisions. Hence, OJ1, the application of mathematics is the main aim for this project. The students work in pairs with the theme "planning a party".

When planning a (imaginary) party, the students have to follow certain restrictions and guidelines:

- A budget of 10,000 DKK.
- The location for the party is a rectangular room, measuring 32 x 8 meters. The rent is 500 DKK. It has a capacity of 80 people.

- There are two types of tables: circular, with a diameter of 128 cm, and square, with a side length of 2 m. Each person needs 80 cm of space.
- You must provide any music or entertainment yourselves.

The students now have to consider the following issues and make decisions based on mathematical arguments:

- Menu: A list of possible starters, main courses, desserts, and drinks is distributed. The list includes prices.
- Seating: Which type of tables are suitable? How many tables are needed for the number of guests? How can the tables be placed in the room?
- Budget: Make a budget for the party based on the number of guests, expenses for the food, room rent, entertainment, etc.
- Summarizing conclusion: Describe the party as a whole using mathematics to explain your choices.

The projects are presented in so-called "café presentations", where groups are placed around the classroom. One student from each group stays at their spot, presenting their work, while the other student circles the room, listening to the other groups' presentations. After a while, they switch. Thereby, everyone gets to present as well as listen. Moreover, the presentation situation is less tense when only a few students are listening, than if the students were to present in front of the whole class.

Chapter 8: Analysis of changes in students' beliefs

This chapter concerns research question 2: *Which changes can be detected in the students' beliefs about mathematics as a discipline after a longitudinal intervention that focuses on developing the students' mathematical overview and judgment?*

To answer this question, I analyze data connected to *changes* in the student's beliefs. Similar to the analysis in chapter 5, I first investigate changes in the beliefs of all students in the two classes by analyzing their responses to the post-questionnaire and comparing them to the pre-questionnaire (section 8.1). In addition to analyzing the students' attitude towards mathematics and their beliefs about the three forms of overview and judgment, I also study their evaluation of the intervention. The section ends with a short summary of the main tendencies.

Subsequently, I investigate the development in the beliefs of the focus students in section 8.2, by analyzing and triangulating data from the questionnaire, the post-interviews (and, when relevant, the midway interviews) and the classroom observations, and comparing the results with those from 2019.

8.1. Data analysis: post-beliefs of all students (post-questionnaire)

Table 17 shows the distribution of the students' answers to each question in the five coding categories. A complete display of the coding criteria for each question can be found in Appendix C, where the number of answers is also displayed along with the numbers from the 2019-questionnaire. As in chapter 5, coding criteria for each aspect of mathematics (attitude, application, etc.) are displayed at the beginning of the respective sections.

Compared to the students' answers to the pre-questionnaire, the average coding value has increased in four of the five aspects of mathematics (see figure 24), indicating a higher level of reflection and/or positive attitude towards mathematics among the students. Only the aspect concerning mathematical activity, which is represented in the drawing task (questions D1 and D2), shows a slight decrease in average coding value. In the following, the students' answers within the five aspects of mathematics are presented and analyzed.

Table 17: Number of answers in the five coding categories (1, 2, 3, 4, and 0) in the post-questionnaire. The columns marked X and Y represent the two classes, with 19 students in each class. Some answers belong in two categories and are thus counted in both, which is why the sum of answers in the five categories might exceed the total number of students (N=38).

Questionnaire 2021. Number of answers in coding categories 1, 2, 3, 4 and 0.

	Code 1 2021			Code 2 2021			Code 3 2021			Code 4 2021			Code 0 2021		
Question/cat.	1X	1Y	**1 Total**	1X	2Y	**2 Total**	3X	3Y	**3 Total**	4X	4Y	**4 Total**	0X	0Y	**0 Total**
1. What do you think of, when you hear the word "mathematics"	6	3	**9**	7	9	**16**	6	7	**13**	1	3	**4**	0	0	**0**
2. How much do you like math in school?	2	0	**2**	7	3	**10**	8	9	**17**	2	7	**9**	0	0	**0**
3. Do you think you are good at math?	0	0	**0**	8	3	**11**	6	14	**20**	5	2	**7**	0	0	**0**
4. What do you like the most about math?	0	0	**0**	4	0	**4**	2	3	**5**	14	16	**30**	0	0	**0**
5. What do you like the least about math?	1	1	**2**	1	2	**3**	1	0	**1**	16	16	**32**	0	0	**0**
A1. Do you think it is important for you to learn math?	1	1	**2**	1	0	**1**	17	18	**35**	0	0	**0**	0	0	**0**
A2. Do you think it is important for everybody in Denmark to lea	5	4	**12**	1	3	**1**	12	11	**23**	1	1	**2**	0	0	**0**
A3. What do you experience and think that math is used for in yo	1	0	**1**	0	0	**0**	10	11	**21**	8	7	**15**	0	0	**0**
A4. What do you experience and think that math is used for in sc	0	0	**0**	4	5	**9**	7	5	**12**	8	11	**19**	0	0	**0**
A5. Do you think that mathematics is more important now than	0	2	**2**	6	3	**9**	3	6	**9**	7	7	**14**	0	1	**1**
B1. How do you imagine that the math, you learn at school, has c	2	2	**4**	0	0	**0**	4	10	**14**	3	5	**8**	10	2	**12**
B2. Why do you think somebody came up with mathematics?	2	3	**5**	0	0	**0**	13	11	**24**	4	5	**9**	0	0	**0**
B3. When do you think mathematics came in to being? (specify y	5	0	**5**	4	11	**15**	9	3	**12**	2	4	**6**	0	1	**1**
B4. What did you learn in school about the history of mathemati	0	0	**0**	4	2	**6**	8	8	**16**	6	8	**14**	1	1	**2**
C1. What is the difference between math and other school subje	3	1	**4**	6	8	**14**	7	8	**15**	4	4	**5**	0	0	**0**
C2. What is the most important in math? Choose max. 3 things:															
C2a. To get a correct result	5	4	**9**										14	15	**29**
C2b. To be able to remember	9	6	**15**										10	13	**23**
C2c. To use the correct methods	8	7	**15**										11	12	**23**
C2d. To come up with your own methods for solution (2019)															
C2e. To understand what the teacher explains	8	4	**12**										11	15	**26**
C2f. To be able to explain what you think	8	12	**20**										11	7	**18**
C2g. To solve problems	4	7	**11**										15	12	**27**
C2h. To know the multiplication tables	3	2	**5**										16	17	**33**
C2i. To be able to find patterns	2	1	**3**										17	18	**35**
C2j. To get good ideas	2	5	**7**										17	14	**31**
C2k. Other	1	1	**2**										18	18	**36**
C2l. To be able to make decisions (2021)	1	7	**8**										18	12	**30**
C3. What is a mathematical problem?	1	4	**5**	7	8	**15**	0	0	**0**	10	7	**17**	0	0	**0**
C4. What do you think a mathematician does?	1	2	**3**	3	2	**5**	8	8	**16**	7	7	**14**	0	0	**0**
X1. What do you remember best from the math lessons in 6th an	2	2	**4**	5	0	**5**	2	8	**10**	10	9	**19**	0	0	**0**
X2. When do you think you have learned most, and why?	2	0	**2**	10	6	**16**	5	8	**13**	2	5	**7**	0	0	**0**
X3. What have been the funniest during math class during the la	4	2	**6**	5	0	**5**	6	8	**14**	4	9	**13**	0	0	**0**
D1. Draw yourself working with math that you find funny	1	0	**1**	7	4	**12**	6	9	**15**	4	1	**5**	2	6	**8**
D2. Explain your drawing	1	0	**1**	7	4	**12**	6	9	**15**	4	1	**5**	2	6	**8**

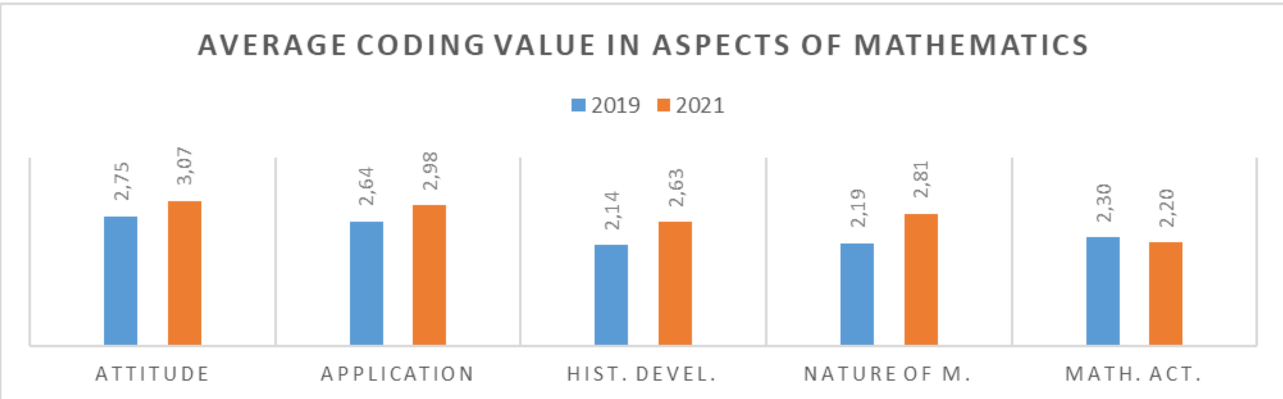


Figure 24: Average coding values in the five aspects of mathematics.

8.1.1. Attitudes towards mathematics

The students' responses to the questions concerning their attitude towards mathematics are coded according to the criteria found in table 18. These criteria are identical to those used in 2019.

Table 18: Coding criteria for questions 1-5: Attitude towards mathematics. Post-questionnaire.

Question / Code	1	2	3	4	0
1. What do you think of when you hear the word "mathematics"?	Emotionally negative ("it is difficult", "I cannot do it", "oh God, not that again", "boo", "oh no"	Content, specific ("multiplication", "fractions")	Content, general ("numbers", "calculating", "my teacher", "calculations")	Emotionally positive ("fun", "challenging")	No answer / misunderstood ("nothing")
2. How much do you like math in school?					No answer
3. Do you think you are good at math?					No answer
4. What do you like the most about math?	Don't know	Emotionally negative ("nothing")	General ("homework", "calculating")	Specific content ("division", "fractions", "math games")	No answer
5. What do you like the least about math?	Don't know	Emotionally negative ("everything")	General ("homework", "calculating")	Specific content ("division", "fractions", "math games")	No answer

Overall, the average coding value of the answers concerning the students' attitude towards mathematics has increased from 2.75 to 3.07 (cf. figure 24). In four out of the five questions, this tendency is clearest in the X-class (cf. table 19), i.e., the students in this class seem to have changed their attitudes the most, especially regarding their likes and dislikes of the subject (questions 4 and 5), and to some extent also regarding their perception of their own performance level. In the Y-class, it is primarily the students' enjoyment of mathematics (question 2) that seems to have increased during the two years of intervention.

Table 19: Difference in average coding value of the questions regarding attitude towards mathematics (1-5).

		Average code value						Diff. in av. code value		
		2019			2021			2019-2021		
	Question/cat.	X 19	Y 19	**Av. 19**	X 21	Y 21	**Av. 21**	Diff. X	Diff. Y	**Diff. all**
1.	**What do you think of, when you hear the word "mathem**	2,08	2,44	**2,26**	2,10	2,45	**2,29**	0,02	0,01	**0,03**
2.	**How much do you like math in school?**	2,46	2,84	**2,65**	2,53	3,21	**2,87**	0,07	0,37	**0,22**
3.	**Do you think you are good at math?**	2,52	2,86	**2,69**	2,84	2,95	**2,89**	0,32	0,08	**0,21**
4.	**What do you like the most about math?**	2,67	3,48	**3,07**	3,50	3,84	**3,67**	0,83	0,37	**0,60**
5.	**What do you like the least about math?**	2,90	3,27	**3,09**	3,68	3,63	**3,66**	0,78	0,36	**0,56**
	Attitude of mathematics (overall, q. 1-5)	2,53	2,98	**2,75**	2,93	3,22	**3,07**	0,40	0,24	**0,32**

A closer study of the numbers behind these averages (table 20) shows that this increase is caused by particularly two tendencies: 1) no students skip any of the questions (code 0), and 2) the students' answers are now more focused on specific mathematical content and less on (negative) emotions.

Table 20: Number of answers in the five coding categories in 2019 (blue numbers) and in 2021 (black numbers), including distribution between the two classes. Questions 1-5.: Attitude towards mathematics. Full table can be found in Appendix D.

Questionnaire 2021.

Number of answers in categories 1, 2, 3, 4 and 0.

Class X: N=21 (2019), N=19 (2021)

Class Y: N=22 (2019), N=19 (2021)

		Code 1						Code 2						Code 3						Code 4						Code 0					
		Code 1 2019			Code 1 2021			Code 2 2019			Code 2 2021			Code 3 2019			Code 3 2021			Code 4 2019			Code 4 2021			Code 0 2019			Code 0 2021		
	Question/cat.	1X	1Y	Tota	1X	1Y	Tota	2X	2Y	Tota	2X	2Y	Tota	3X	3Y	Tota	3X	3Y	Tota	4X	4Y	Tota	4X	4Y	Tota	0X	0Y	Tota	0X	0Y	Tota
1.	**What do you think of, when you**	5	7	**12**	6	3	**9**	8	4	**12**	7	9	**16**	9	10	**19**	6	7	**13**	1	4	**5**	1	3	**4**	2	0	**2**	0	0	**0**
2.	**How much do you like math in sc**	4	1	**5**	2	0	**2**	6	7	**13**	7	3	**10**	13	12	**25**	8	9	**17**	1	5	**6**	2	7	**9**	0	0	**0**	0	0	**0**
3.	**Do you think you are good at ma**	3	3	**6**	0	0	**0**	6	3	**9**	8	3	**11**	13	10	**23**	6	14	**20**	1	6	**7**	5	2	**7**	0	0	**0**	0	0	**0**
4.	**What do you like the most about**	3	0	**3**	0	0	**0**	4	3	**7**	4	0	**4**	3	1	**4**	2	3	**5**	9	16	**25**	14	16	**30**	2	1	**3**	0	0	**0**
5.	**What do you like the least about**	1	1	**2**	1	1	**2**	4	4	**8**	1	2	**3**	4	1	**5**	1	0	**1**	10	15	**25**	16	16	**32**	2	1	**3**	0	0	**0**

The second tendency is seen in all three questions, where the emotional aspect is included in the coding criteria (1, 4, and 5). In question 1, there is a slight decrease in the number of emotionally loaded answers (both negative and positive). The Y-class shows the largest change, with only three students associating mathematics with negative emotions compared to seven in 2019. Also, the positive emotional indications have decreased in this class (from 4 to 3). In contrast to the 2019-results, the students in both classes apparently associate mathematics with specific content (16 students compared to 12 in 2019) to a higher degree than general content (13 students, 19 in 2019). This tendency is also more apparent in class Y.

When asked what they like the most and least in mathematics (questions 4 and 5), the students, likewise, give responses related to specific content (code 4) rather than general content (code 3) or emotions (code 2). In both questions, this movement is more explicit in the X-class; however, there still seems to be more students here with negative emotions towards mathematics than in the Y-class. When it comes to the students' enjoyment of mathematics (question 2), the distribution of answers is quite similar to the results in 2019. However, there is a slight movement from the low to the high coding values, as only two students mark the unhappiest emoticon (five students in 2019), and nine students mark the happiest emoticon, compared to six in 2019. As in 2019, the two middle emoticons are still the most popular, the happier one marked by 17 students (25 in 2019, including six students putting their mark in the middle), and the slightly unhappy emoticon marked by 10 students (13 in 2019, also including the six students marking the middle).

Also, regarding the students' perception of their own performance level (question 3), the majority of the students marked one of the two emoticons in the middle. This is similar to 2019. However, compared to 2019, where six students marked the unhappiest emoticon, no students choose this option in 2021. The number of students marking the happiest emoticon is unchanged (seven students) when looking at the two classes together. Nevertheless, five of these students are from the X-class and two are from the Y class, where the distribution was one in the X-class and six in the Y-class in 2019. Among the students in the Y-class there thus seems to be a movement from the extremes towards the middle options. In the X-class, the change is even larger, and the movement is different. The number of markings of the slightly unhappy emoticon has doubled (from four to eight), and the number of markings of the slightly happy emoticon has gone from 11 to six. Thereby, the movement in the X-class seems to be from code 3 to either code 2 or code 4.

8.1.2. Beliefs about the application of mathematics

Questions A1-A5 concern the application of mathematics and are coded according to the criteria in table 21, which have not changed from the 2019-criteria.

Table 21: Coding criteria for questions A1-A5: Application of mathematics. Post-questionnaire

Question / Code	1	2	3	4	0
A1. Do you think it is important for you to learn math? Yes, because: / No, because:	No	Yes Unspecified justification or vague (”it is important”, ”you need it”)	Yes Generally specified (”to get an education”, ”to get a job”, ”you use it for everything”)	Yes Further specified (”I’m going to be an engineer”)	No answer
A2. Do you think it is important for everybody in Denmark to learn math? Yes, because: / No, because:	No	Yes Unspecified justification or vague (”it is important”, ”you need it”)	Yes Generally specified (”to get an education”, ”to get a job”, ”you use it for everything”)	Yes Further specified (”I’m going to be an engineer”)	No answer
A3. What do you experience and think that math is used for in your daily life?	Nothing / don’t know	Not specified (”everything”)	Shopping + possibly something else	Several things	No answer / misunderstood
A4. What do you experience and think that math is used for in society?	Nothing / don’t know	Everyday things (”Shopping”)	General, society (”everything”, ”work”, ”money”)	Specific, society (”to build roads”, ”to pay tax”)	No answer / misunderstood
A5. Do you think that mathematics is more important now than 100 years ago? Why / why not?	More important 100 years ago OR equally important, no or vague justification	More important now, no or vague justification	More important 100 years ago with justification	More important now with justification	No answer / misunderstood / don’t know

As illustrated in figure 24 and table 22, the students’ answers concerning the application aspect of mathematics also increased in average coding value from 2.64 to 2.98. This increase is strongest in questions A4 and A5 when looking at the two classes as a whole. There lie, however, quite large differences between the two classes behind this average. Table 22 shows that where the average value in both classes has changed in question A4 concerning the application of mathematics in society (from 2.35 to 3.25), only the X-class has increased the average coding value rather notably in question A5 (from 2.00 to 3.06), which concerns the importance of mathematics now compared to 100 years ago. Only in one question (A2) has the average coding value decreased. Yet it has done only slightly (from 2.63 to 2.47).

Table 22: Difference in average coding value of the questions regarding application of mathematics (A1-A5).

		Average code value						Diff. in av. code value		
		2019			2021			2019-2021		
	Question/cat.	X 19	Y 19	Av. 19	X 21	Y 21	Av. 21	Diff. X	Diff. Y	Diff. all
A1.	Do you think it is important for you to learn math?	2,76	2,82	2,79	2,84	2,89	2,87	0,08	0,08	0,08
A2.	Do you think it is important for everybody in Denmark to	2,76	2,50	2,63	2,47	2,47	2,47	-0,29	-0,03	-0,15
A3.	What do you experience and think that math is used for i	3,14	2,95	3,05	3,32	3,39	3,35	0,17	0,43	0,30
A4.	What do you experience and think that math is used for i	2,19	2,50	2,35	3,21	3,29	3,25	1,02	0,79	0,90
A5.	Do you think that mathematics is more important now th	2,00	2,75	2,40	3,06	2,84	2,94	1,06	0,09	0,54
	Application of mathematics (overall, q. A1-A5)	2,57	2,70	2,64	2,98	2,98	2,98	0,41	0,27	0,33

Table 23 shows the number of answers assigned the five coding values in the two classes, in 2019 (blue numbers) and in 2021.

Table 23: Number of answers in the five coding categories in 2019 (blue numbers) and in 2021 (black numbers), including distribution between the two classes. Questions A1-A5: Application of mathematics. Full table can be found in Appendix D.

Questionnaire 2021.

Number of answers in categories 1, 2, 3, 4 and 0.

Class X: N=21 (2019), N=19 (2021)

Class Y: N=22 (2019), N=19 (2021)

		Code 1						Code 2					
		Code 1 2019			Code 1 2021			Code 2 2019			Code 2 2021		
	Question/cat.	1X	1Y	Tota	1X	1Y	Tota	2X	2Y	Tota	2X	2Y	Tota
A1.	Do you think it is important for y	2	1	3	1	1	2	2	6	8	1	0	1
A2.	Do you think it is important for e	3	3	6	5	4	9	1	7	8	1	3	4
A3.	What do you experience and thir	1	2	3	1	0	1	1	2	3	0	0	0
A4.	What do you experience and thir	7	4	11	0	0	0	5	6	11	4	5	9
A5.	Do you think that mathematics is	6	6	12	0	2	2	2	1	3	6	3	9

		Code 3						Code 4						Code 0					
		Code 3 2019			Code 3 2021			Code 4 2019			Code 4 2021			Code 0 2019			Code 0 2021		
	Question/cat.	3X	3Y	Tota	3X	3Y	Tota	4X	4Y	Tota	4X	4Y	Tota	0X	0Y	Tota	0X	0Y	Tota
A1.	Do you think it is important for y	16	11	27	17	18	35	1	4	5	0	0	0	0	0	0	0	0	0
A2.	Do you think it is important for e	15	10	25	12	11	23	2	2	4	1	1	2	0	0	0	0	0	0
A3.	What do you experience and thir	13	13	26	10	11	21	6	5	11	8	7	15	0	0	0	0	0	0
A4.	What do you experience and thir	7	5	12	7	5	12	2	6	8	8	11	19	0	1	1	0	0	0
A5.	Do you think that mathematics is	0	6	6	3	6	9	8	10	18	7	7	14	5	1	6	0	1	1

The students' answers concerning the importance of learning mathematics (questions A1 and A2), overall indicate that the students have become more aware of this issue, as considerably fewer of the answers are assigned code 2 ("yes", unspecified or vague justification). In 2019, eight answers were assigned this code in each question; this number has now decreased to one in A1, and four in A2.

36 of the 38 students find it important for themselves to learn mathematics, and 35 of them give a generally specified explanation for this (e.g., "to get an education", "you use it for everything", or "I will need it in my future life"), which is an increase from 27 in 2019. None of them give a more specific or personal explanation (code 4). Only two students (three in 2019) answer "no", both of them without any explanation. However, when it comes to the general importance of learning mathematics (question A2), the number of students answering "no" has increased, from six in 2019 to nine in 2021. They do, though, have an explanation for this, as most of them argue that some people will not need it for their job or do

not like it. The students answering “yes” to question A2 give – like to A1 – a generally specified answer, e.g., that people need it for their jobs or in their daily lives.

The students’ answers concerning the use of mathematics in their daily lives (question A3), have not changed remarkably. The majority (21 students) still see shopping as the primary use of mathematics. Yet, the small change in the number of answers assigned code 4 (mentioning several things, from 11 in 2019 to 15 in 2021), indicates that more students are now able to give various examples of what mathematics might be used for apart from shopping. This includes baking, embroidery, statistics, calculating time, and playing games. Furthermore, only one student is not able to give concrete examples (code 1 and 2), this number being six in 2019. The distribution of answers to question A3 and the change in distribution is almost equal in the two classes.

The tendency from A3 is even more clear in question A4 concerning the use of mathematics in society, where the number of students not able to answer has gone from 11 to none. The number of students naming everyday things or giving more general answers, like “everything” or “work”, is almost unchanged (23 in 2019, 21 in 2021). In contrast, there are 19 students that give specific examples of the use of mathematics in society, compared to only 8 in 2019. Hence, this number has more than doubled. Their answers include such diverse examples as building houses and roads, calculating votes at elections, economy, taxes, rocket science, conducting surveys, and making decisions. As seen in table 23, this question thus has the largest change in average coding value.

In question A5, the students are asked if they think that mathematics is more important now than 100 years ago. As in 2019, most students (14) justify that they find mathematics more important now, but this is a decrease from 18 students in 2019. In contrast, the number of students who do *not* justify that mathematics is more important now has gone from three to nine. Students that find mathematics less or equally important now has also decreased, but more of them are now able to justify their answer: Nine students give a reason, compared to six in 2019, and only two students do not. This number was 12 in 2019.

The change in the students’ answers to this question is far more substantial in the X-class, exemplified in the fact that no answers are assigned code 0 (five in 2019), and none of the students are unable to justify why they find mathematics less or equally important now (six in 2019).

Overall, in this category of questions, the differences between the two classes in 2019 has more or less equalized in 2022. With a few exceptions, the number of answers within each coding category is almost the same.

8.1.3. Beliefs about the historical development of mathematics

Table 24 shows the criteria for the coding of the students' answers about the historical development of mathematics.

Table 24: Coding criteria for questions B1-B5: Historical development of mathematics. Post-questionnaire.

Question	Code	1	2	3	4	0
B1. How do you imagine that the math you learn at school has come into being?		Don't know or deficient response	Emotionally negative	Vague justification ("somebody just thought of it" or "to measure something")	Reflected response ("it has always been part of life", "someone felt a need for counting and calculating")	No answer / misunderstood
B2. Why do you think somebody came up with mathematics?		Don't know	Emotionally negative	Vague justification ("to make it easier", "to add 2 and 2")	Reflected response (To be able to be more precise, e.g., when building something", "because we need units to the things we do"	No answer / misunderstood
B3. When do you think mathematics came in to being? (specify year or period)		Don't know	Specific (random) year ("1773", "1900")	Period (random) ("the Iron Age")	Justified time indication ("since humans came into being")	No answer / misunderstood
B4. What did you learn in school about the history of mathematics?		Nothing	Don't remember	Vague specification	Precise specification	No answer / misunderstood

The answers to three out of four questions have increased in average coding value (table 25).

Table 25: Difference in average coding value of the questions regarding the historical development of math (B1-B4).

		Average code value						Diff. in av. code value		
		2019			2021			2019-2021		
	Question/cat.	X 19	Y 19	Av. 19	X 21	Y 21	Av. 21	Diff. X	Diff. Y	Diff. all
B1.	How do you imagine that the math, you learn at school, I	2,19	2,00	2,09	1,37	2,74	2,05	-0,82	0,74	-0,04
B2.	Why do you think somebody came up with mathematics?	2,57	2,64	2,60	3,00	2,95	2,97	0,43	0,31	0,37
B3.	When do you think mathematics came in to being? (speci	2,00	2,14	2,07	2,40	2,47	2,44	0,40	0,34	0,37
B4.	What did you learn in school about the history of mathe	2,15	1,50	1,81	2,95	3,16	3,05	0,80	1,66	1,24
	Historical development of mathematics (overall, q.B1-B4)	2,23	2,07	2,14	2,43	2,83	2,63	0,20	0,76	0,48

Two of the questions stand out: There seems to be a large difference between the two classes in question B1 ("How do you imagine that the math you learn at school has come into being?"); and the increase is noteworthy in question B4 ("What did you learn in school about the history of mathematics?"), where the average coding value has gone from 1.81 in 2019 to 3.05 in 2021. However, the latter is most likely affected by the change in the phrasing of question B4. In the following, the students' answers are elaborated. Table 26 shows the number of answers assigned each of the coding values.

Table 26: Number of answers in the five coding categories in 2019 (blue numbers) and in 2021 (black numbers), including distribution between the two classes. Questions B1-B5: historical development of mathematics. Full table can be found in Appendix D.

Questionnaire 2021.

Number of answers in categories 1, 2, 3, 4 and 0.

Class X: N=21 (2019), N=19 (2021)

Class Y: N=22 (2019), N=19 (2021)

		Code 1						Code 2						Code 3						Code 4						Code 0					
		Code 1 2019			Code 1 2021			Code 2 2019			Code 2 2021			Code 3 2019			Code 3 2021			Code 4 2019			Code 4 2021			Code 0 2019			Code 0 2021		
	Question/cat.	1X	1Y	Tot	1X	1Y	Tot	2X	2Y	Tot	2X	2Y	Tot	3X	3Y	Tot	3X	3Y	Tot	4X	4Y	Tot	4X	4Y	Tot	0X	0Y	Tot	0X	0Y	Tota
B1.	How do you imagine that the math, yo	6	7	**13**	2	2	**4**	2	0	**2**	0	0	**0**	8	7	**15**	4	10	**14**	3	4	**7**	3	5	**8**	2	4	**6**	10	2	**12**
B2.	Why do you think somebody came up	3	0	**3**	2	3	**5**	4	1	**5**	0	0	**0**	9	12	**21**	13	11	**24**	4	5	**9**	4	5	**9**	1	4	**5**	0	0	**0**
B3.	When do you think mathematics came	1	3	**4**	5	0	**5**	14	14	**28**	4	11	**15**	3	4	**7**	9	3	**12**	1	1	**2**	2	4	**6**	2	0	**2**	0	1	**1**
B4.	What did you learn in school about the	9	18	**27**	0	0	**0**	0	0	**0**	4	2	**6**	2	1	**3**	8	8	**16**	7	3	**10**	6	8	**14**	2	0	**2**	1	1	**2**

When asked about the origin of the mathematics taught in school (question B1), the number of students giving a reflected response (eight) is almost the same as in 2019 (seven), as is the number of students, who give a vague justification for their answer (14 compared to 15 in 2019). However, this question is the only one of the B-questions with a drop in the average coding value. The numbers in table 26 show that this is connected to the number of students, who misunderstand or skip the question. This number has doubled (from six in 2019 to twelve in 2021). Looking through these students' answers, it shows that none of them skip the question, but that the coding value 0 is assigned because they misunderstood it. A closer look at the students' answers shows that some of them seem to have misread the question, as they respond to *what* they imagine mathematics has become, instead of *how* it came into being. Others respond in a way that indicates that they perceive the question as related to their opinion about mathematics (e.g., "It has become more fun", or "I guess it is okay"), and a few give answers about how they learned mathematics ("I learned it from a teacher"). Thereby, the misunderstandings overall appear to be caused by incorrect reading of the question. Of these twelve students, 10 are from the X-class, which explains the large difference in the change in average coding value between the two classes. However, far less students answer, "I don't know" (four, compared to 13 in 2019).

In relation to the students' idea of why mathematics came into being (question B2), the majority of the students (24, compared to 21 in 2019) still give a vague justification for their answer to this question (e.g., "to count coins"), and the number of reflected answers is unchanged (nine). In contrast to the previous question, the number of students who skip or misunderstand the question or give emotionally negative answers have, in both cases, gone from five to none.

The phrasing of question B3, which might have affected the students' answers in 2019, has remained the same for the sake of comparison (cf. section 6.1). This time though, the number of students responding with a specific and apparently random year has decreased from 28 to 15, which is particularly evident in the Y-class with a drop from 14 to four. In contrast, the numbers of both justified responses (six students) and responses indicating a time period (12 students) have increased from two and seven, respectively.

Conversely, the phrasing of question B4 is different from 2019, as it addresses *what* the students have learned about the history of mathematics, instead of *if* they have learned anything about this. Thereby, the coding criteria have also changed, which makes it difficult to compare the numbers in the two questionnaires. However, it is noticeable that no students answer “nothing” and only six students answer “I don’t remember” to this question. 16 students give a vague specification, but 14 students are able to name specific teaching topics related to the history of mathematics; most of them mention Pythagoras or Euclid.

8.1.4. Beliefs about the nature of mathematics as a subject area

Table 27: Coding criteria for questions C1-C4: Nature of mathematics as a subject area. Post-questionnaire.

Question	Code	1	2	3	4	0
C1. What is the difference between math and other school subjects?		Don't know / no difference / deficient response	Emotional response ("I don't like math", "I like English better")	Content ("There are more numbers in math", "in math you calculate")	Working methods ("we work more with computer", "there is more group work")	No answer / misunderstood
C2. What is the most important in math? Choose max. 3 things:						
C2a. To get a correct result C2b. To know rules and formulas by heart C2c. To use the correct methods C2d. - C2e. To understand what the teacher explains C2f. To be able to explain what you think C2g. To solve problems C2h. To know the multiplication tables C2i. To be able to find patterns C2j. To get good ideas C2k. Other:___ C2l. To be able to make decisions		Marked				Not marked
C3. What is a mathematical problem?		Don't know	Vague ("a problem with math")	Backwards understanding ("when you calculate wrong")	Sign of understanding	No answer / misunderstood
C4. What do you think a mathematician does?		Don't know	Wrong or irrelevant response.	Vague response ("does math", "calculates")	Sign of understanding ("works with mathematical problems")	No answer / misunderstood

Table 28: Difference in average coding value of the questions regarding the nature of mathematics as a subject (C1, C3 and C4). Question 2 is omitted from this calculation, as it does not follow the same coding procedure.

	Average code value						Diff. in av. code value		
	2019			2021			2019-2021		
Question/cat.	X 19	Y 19	**Av. 19**	X 21	Y 21	**Av. 21**	Diff. X	Diff. Y	**Diff. all**
What is the difference between math and other school su	2,27	2,78	**2,53**	2,60	2,71	**2,55**	0,33	-0,07	**0,02**
What is a mathematical problem?	1,14	1,41	**1,28**	3,06	2,53	**2,78**	1,91	1,12	**1,50**
What do you think a mathematician does?	2,81	2,45	**2,76**	3,11	3,05	**3,08**	0,30	0,60	**0,32**
Nature of mathematics (overall, q. C1+C3+C4)	2,08	2,22	**2,19**	2,92	2,76	**2,81**	0,85	0,55	**0,62**

The average coding values for the questions concerning the nature of mathematics as a subject area (table 28) are only calculated for the questions C1, C3 and C4, as question C2 follows a different coding procedure (cf. section 4.2.2). As seen in relation to the analysis of the other aspects, there is an overall increase in the average coding value in these questions too. This especially applies to question C3, going from 1.28 to 2.78, indicating that the students are more aware of what a mathematical problem is. This increase is particularly strong in the X-class.

The increase is far less substantial in question C1, concerning the difference between mathematics and other school subjects, and in the Y-class the average coding value has actually decreased a little. The numbers in table 29 reveal that the numbers for the Y-class are almost unchanged for code 1, 4 and 0, but there are more answers assigned code 2 (eight, compared to five in 2019), which indicates an emotional answer, and less answers assigned code 3 (eight, compared to 12 in 2019), indicating mathematical content. It, thereby, seems that more students in the Y-class express their preferences in their answer. However, the emotional expression can be either negative or positive, and a scan of the answers show that only three of the 12 answers are negative. The rest include words like "fun", "important" or "better". In contrast, four out of the six emotionally loaded responses in the X-class are negative, using words such as "boring" or "difficult". In the X-class, there is, furthermore, an increase from one to four in answers that see working methods as the difference between mathematics and other school subjects.

Table 29: Number of answers in the five coding categories in 2019 (blue numbers) and in 2021 (black numbers), including distribution between the two classes. Questions C1-C4: Nature of mathematics as a subject area. Full table can be found in Appendix D.

Questionnaire 2021.

Number of answers in categories 1, 2, 3, 4 and 0.

Class X: N=21 (2019), N=19 (2021)

Class Y: N=22 (2019), N=19 (2021)

		Code 1						Code 2						Code 3						Code 4						Code 0					
		Code 1 2019			Code 1 2021			Code 2 2019			Code 2 2021			Code 3 2019			Code 3 2021			Code 4 2019			Code 4 2021			Code 0 2019			Code 0 2021		
	Question/cat.	1X	1Y	Tot	1X	1Y	Tot	2X	2Y	Tot	2X	2Y	Tot	3X	3Y	Tot	3X	3Y	Tot	4X	4Y	Tot	4X	4Y	Tot	0X	0Y	Tot	0X	0Y	Tot
C1.	**What is the difference between mat**	5	2	**7**	3	1	**4**	7	5	**12**	6	8	**14**	9	12	**21**	7	8	**15**	1	4	**5**	4	4	**5**	0	0	**0**	0	0	**0**
C2.	**What is the most important in math**																														
C2a.	***To get a correct result***	9	6	**15**	5	4	**9**																			12	16	**27**	14	15	**29**
C2b.	***To be able to remember***	7	1	**8**	9	6	**15**																			14	21	**35**	10	13	**23**
C2c.	***To use the correct methods***	5	7	**12**	8	7	**15**																			16	15	**31**	11	12	**23**
C2d.	***To come up with your own methods***	10	12	**22**																						11	10	**21**			
C2e.	***To understand what the teacher med***	9	8	**17**	8	4	**12**																			12	14	**26**	11	15	**26**
C2f.	***To be able to explain what you think***	6	10	**16**	8	12	**20**																			15	12	**27**	11	7	**18**
C2g.	***To solve problems***	4	2	**6**	4	7	**11**																			17	20	**37**	15	12	**27**
C2h.	***To know the multiplication tables***	8	6	**14**	3	2	**5**																			13	16	**29**	16	17	**33**
C2i.	***To be able to find patterns***	0	1	**1**	2	1	**3**																			21	21	**42**	17	18	**35**
C2j.	***To get good ideas***	2	6	**8**	2	5	**7**																			19	16	**35**	17	14	**31**
C2k.	***Other***	1	3	**4**	1	1	**2**																			20	19	**39**	18	18	**36**
C2l.	***To be able to make decisions (only 2(***				1	7	**8**																						18	12	**30**
C3.	**What is a mathematical problem?**	14	13	**27**	1	4	**5**	3	7	**10**	7	8	**15**	0	0	**0**	0	0	**0**	1	1	**2**	10	7	**17**	3	1	**4**	0	0	**0**
C4.	**What do you think a mathematicia**	1	4	**5**	1	2	**3**	1	1	**2**	3	2	**5**	12	12	**24**	8	8	**16**	5	3	**8**	7	7	**14**	2	2	**2**	0	0	**0**

When the options in question C2 ("What is the most important in math") are divided into a dualistic and a relativistic perspective of mathematics (table 30), the result resembles what was seen in the 2019 questionnaire. At that time, the distribution was 67-55, in favor of the dualistic options. Now, the numbers are 56-50, which means that the majority still choose the options belonging to the dualistic end of the spectrum.

Table 30: Options in question C2, categorized according to the spectrum of dualistic/relativistic perspective on mathematics, post-questionnaire. Numbers indicate markings, the numbers from 2019 are in parentheses.

Dualistic:	**Relativistic:**
a) To get a correct result *9 (15)* b) To know rules and formulas by heart *15 (8)* c) To use the correct methods *15 (12)* e) To understand what the teacher means 12 *(17)* h) To know the multiplication tables *5 (14)* k) Other: *0 (1)*	d) To come up with your own methods for a solution (*not an option in 2021) (22)* f) To be able to explain what you think *20 (16)* g) To solve problems *11 (6)* i) To be able to find patterns *3 (1)* j) To get good ideas *7 (8)* l) To be able to make decisions *8 (not an option in 2019)* k) Other: "Understanding why" *1 (2)*
Undetermined: k) Other: "to get a job and make money and not become homeless" k) Other: "to be able to do all that you need it for"	

There is, however, a change in the preferred options. For example, the most popular option in 2019 (option d) is not part of the questionnaire in 2021. The option chosen most often (20) is now option f): "To be able to explain what you think", which is an increase from 16 in 2019. Also, options g): "To solve problems" and i): "To be able to find patterns" have increased in popularity, but so have options b): "To know rules and formulas by heart", and c): "To use the correct methods". The added option l): "To be able to make decisions" is chosen by eight students, whereof seven are from the Y-class. The largest decrease is seen in option h): "To know the multiplication tables", which drops from 14 to five markings. The reason for this decrease might be that there is not as much emphasis on memorizing multiplication tables in 7th grade compared to 5th or 6th grade.

As mentioned, the average coding value for question C3 ("What is a mathematical problem?") has increased quite a lot from 2019. Actually, this question has the largest difference in average coding value between the two questionnaires, which is clearest in the X-class, where the average coding value has gone from 1.14 to 3.06. The number of students answering, "I don't know" has decreased from 27 to five (whereof only one is from the X-class), thereby giving a clear sign that the students have become aware of this concept during the two years of intervention. Furthermore, no students skip this question, compared to four in 2019. The number of students that show signs of understanding in their answers (e.g., "when there is a problem in the world that we can solve with mathematics" or "finding the volume of a cube, for example") has increased from two to seventeen, ten of them from the X-class.

The same tendency is seen in question C4 ("What do you think a mathematician does?), where more students show signs of understanding (14, compared to eight in 2019), and less students give vague responses or are unable to answer.

8.1.5. Mathematical activity

The coding criteria for questions D1 and D2 are similar (table 31), as D2 is a written explanation of the drawing in D1.

Table 31: Coding criteria for questions D1-D2: Mathematical activity (drawing). Post-questionnaire.

Question	Code 1	2	3	4	0
D1. Draw yourself working with mathematics that you find fun:	Emotionally negative	Traditional student role (alone, sitting by a desk, solving tasks)	Other, but in a school setting	Non-school setting	No answer / misunderstood
D2 Explain your drawing	Emotionally negative	Traditional student role (alone, sitting by a desk, solving tasks)	Other, but in a school setting	Non-school setting	No answer / misunderstood

The average coding value has increased slightly for the X-class but decreased for the Y-class (table 32).

Table 32: Difference in average coding value of the questions regarding mathematical activity (drawing) (D1-D2).

		Average code value						Diff. in av. code value		
		2019			2021			2019-2021		
	Question/cat.	X 19	Y 19	Av. 19	X 21	Y 21	Av. 21	Diff. X	Diff. Y	Diff. all
D1.	Draw yourself working with mathematics that you find	2,29	2,32	2,30	2,45	1,95	2,20	0,16	-0,37	-0,11
D2.	Explain your drawing	2,29	2,32	2,30	2,45	1,95	2,20	0,16	-0,37	-0,11
	Mathematical activity (overall, q. D1-D2)	2,29	2,32	2,30	2,45	1,95	2,20	0,16	-0,37	-0,11

An analysis of the students' answers provides a possible explanation for this. The drawing task is the only question that is not answered digitally. Therefore, the students' received a piece of paper for this response. Unfortunately, eight students did either not finish the task in time or did not hand it in. Thus, there is a high number of answers assigned the coding value 0, most of them from class Y.

Table 33: Number of answers in the five coding categories in 2019 (blue numbers) and in 2021 (black numbers), including distribution between the two classes. Questions D1-D5: Mathematical activity (Drawing). Full table can be found in Appendix D.

Questionnaire 2021.
Number of answers in categories 1, 2, 3, 4 and 0.
Class X: N=21 (2019), N=19 (2021)
Class Y: N=22 (2019), N=19 (2021)

		Code 1						Code 2						Code 3						Code 4						Code 0					
		Code 1 2019			Code 1 2021			Code 2 2019			Code 2 2021			Code 3 2019			Code 3 2021			Code 4 2019			Code 4 2021			Code 0 2019			Code 0 2021		
	Question/cat.	1X	1Y	Tota	1X	1Y	Tota	2X	2Y	Tota	2X	2Y	Tota	3X	3Y	Tota	3X	3Y	Tota	4X	4Y	Tota	4X	4Y	Tota	0X	0Y	Tota	0X	0Y	Tota
D1.	Draw yourself working with mat	2	0	2	1	0	1	8	14	22	7	4	12	6	5	11	6	9	15	3	2	5	4	1	5	2	1	3	2	6	8
D2.	Explain your drawing	2	0	2	1	0	1	8	14	22	7	4	12	6	5	11	6	9	15	3	2	5	4	1	5	2	1	3	2	6	8

The number of students drawing themselves in situations outside a school context is unchanged (see table 33). However, the number of students drawing themselves in a traditional student role, sitting alone at a desk, has decreased considerably from 22 to 12, and the number of students drawing themselves in other school contexts has in contrast increased from 11 to 15. This tendency is also strongest in the Y-class. When looking through the drawings, it is clear that many of the drawings depict situations from the intervention that they have enjoyed. For example, the pancake activity (described in section 7.4.3) is the motive for several drawings. Another example is presenting results from some of the projects included in the intervention.

8.1.6. Evaluation of intervention

The last three questions in the post-questionnaire are an evaluation of the intervention, and the answers are, as previously described, coded according to the level of specification and connection to the intervention (table 34).

Table 34: Coding criteria for questions E1-E3: Evaluation of intervention. Post-questionnaire.

Question	Code	1	2	3	4	0
E1. What do you remember best from the mathematics classes in 6th and 7th grade?		Don't know / can't remember / nothing	Not specified / not related to mathematics	Specified, not related to intervention	Specified, related to intervention	No answer / misunderstood
E2. When do you think you learned most, and why?						
E3. What has been the most fun in mathematics class in the last two years?						

The average coding values are above 2.50 for all three questions, and thus in the higher half of the spectrum, not counting code 0 (table 35).

Table 35: Average coding value of the questions regarding Evaluation of the intervention (E1-E3).

		Average code value		
		2021		
	Question/cat.	**X 21**	**Y 21**	**Av. 21**
E1.	**What do you remember best from the math lessons in 6th and 7th grade?**	3,05	3,26	**3,16**
E2.	**When do you think you have learned most, and why?**	2,37	2,95	**2,66**
E3.	**What have been the funniest during math class during the last two years?**	2,53	3,26	**2,89**
	Evaluation of intervention (overall, q. E1-E3)	2,65	3,16	**2,90**

The distribution of answers (table 36) shows that half of the students (19) mention examples connected to the intervention when asked what they remember best from the mathematics lesson in 6th and 7th grade. The examples most commonly mentioned are "Pythagoras" and "Euclid", which are mentioned eleven and five times respectively, but also "triangles", "projects" and "pancakes" are mentioned. Of the other half of the students, ten give specified and relevant examples not directly connected to the intervention (e.g., fractions, probability, or geometry)

Table 36: Number of answers in the five coding categories in 2021, including distribution between the two classes. Questions E1-E3: Evaluation of intervention. Full table can be found in Appendix D.

	Questionnaire 2021.																														
	Number of answers in categories 1, 2, 3, 4 and 0.																														
	Class X: N=21 (2019), N=19 (2021)																														
	Class Y: N=22 (2019), N=19 (2021)																														
		Code 1						**Code 2**						**Code 3**						**Code 4**						**Code 0**					
		Code 1 2019			Code 1 2021			Code 2 2019			Code 2 2021			Code 3 2019			Code 3 2021			Code 4 2019			Code 4 2021			Code 0 2019			Code 0 2021		
	Question/cat.	1X	1Y	Tota	1X	1Y	Tota	2X	2Y	Tota	2X	2Y	Tota	3X	3Y	Tota	3X	3Y	Tota	4X	4Y	Tota	4X	4Y	Tota	0X	0Y	Tota	0X	0Y	Tota
E1.	**What do you remember best from t**	-	-	-	2	2	**4**	-	-	-	5	0	**5**	-	-	-	2	8	**10**	-	-	-	10	9	**19**	-	-	-	0	0	**0**
E2.	**When do you think you have learne**	-	-	-	2	0	**2**	-	-	-	10	6	**16**	-	-	-	5	8	**13**	-	-	-	2	5	**7**	-	-	-	0	0	**0**
E3.	**What have been the funniest durin**	-	-	-	4	2	**6**	-	-	-	5	0	**5**	-	-	-	6	8	**14**	-	-	-	4	9	**13**	-	-	-	0	0	**0**

When asked when they have learned the most during the last two years, only seven students mention examples that are directly related to the intervention, but 13 students give other examples of mathematical activities. These activities or topics normally occur in the curriculum. However, as they have been part of the mathematics teaching during the intervention, they have been taught while implementing the principles, and can thus be said to be indirectly connected to the intervention. There is quite a large number of students who give an answer that is not related to mathematics, which might be because of the phrasing of the question that does not actually mention mathematics. Hence, many answers concern general learning situations.

In question E3, the students are asked what they have found most fun in the mathematics lessons during the intervention. 13 students mention examples directly connected to the intervention (e.g., "outside activities", "doing presentations", "number of the day"[8] or "learning about the old mathematicians"). An almost equal number of answers include other mathematics-related examples (e.g., "math games" or "functions").

8.1.7. Overall development in the students' post-beliefs about mathematics as a discipline

The analysis of the post-questionnaire and the comparison to 2019 give the general impression that the students' beliefs about mathematics as a discipline have developed in the direction of a more positive attitude and an increased level of reflection. Throughout all aspects, there are close to no answers assigned code 0. The students thereby seem less doubtful or insecure about the topics in the questions, indicating that the intervention has made them aware of aspects connected to mathematics as a discipline. This is, for example, very clear in questions C3 and C4 about the nature of mathematics as a subject area. The amount of emotionally negative answers has, likewise, decreased considerably and been replaced by more content-related responses. Overall, the students' answers include more

[8] The teacher regularly introduced a lesson with "number of the day". In this activity, a number (e.g. a fraction), or, if relevant, a mathematical concept, is written on the whiteboard. The students are then asked to represent this number in four different ways: on a number line, in a text, with a drawing, and as part of a calculation.

examples (e.g., of the application and the historical development of mathematics, many of which have been part of the intervention). Several responses include a higher level of reflection, and the students seem to be better at justifying their answers.

The development becomes very clear when the sum of the coding values is calculated for each student and categorized into the three levels that were used in the process of selecting focus students (cf. section 5.3.1). The red level represents a rather low level of reflection and/or a negative attitude towards mathematics, with a sum of coding values below 41. The yellow level is a middle level with a total sum between 41 and 49, and the green level represents a high level with a sum of 50 or more.

A comparison of the distribution of the three levels in 2019 and 2021 (figure 25) shows an extensive movement towards the green level, with an average increase of approximately 10 points in the coding value sum (table 37).

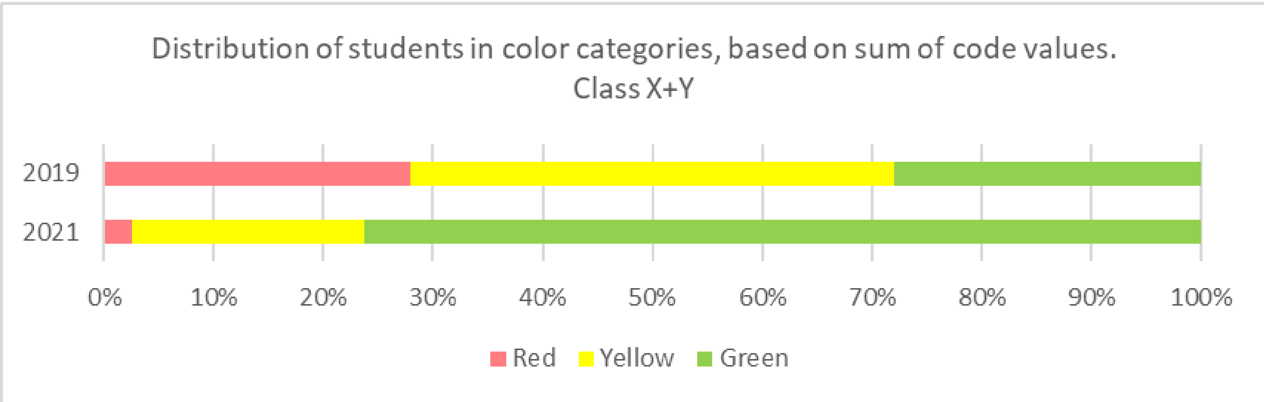


Figure 25: Comparison of the distribution of students in the three levels of code value sum in 2019 and 2021

Where the distribution was almost equal in 2019, there is now only one student in the red level, 8 students in the yellow level, and 29 students in the green level. Moreover, 22 out of the 29 students in the green level have a coding value sum above 55 compared to only three in 2019 (table 38).

Table 37: Categorizing of all students in the two classes according to the sum of coding values. Red: <41, yellow: 41-49, green: >49.

Class X, code values (students)			
Name	**Class**	**Sum 2019**	**Sum 2021**
[redacted]	X	37	44
Molly	X	**56**	**61**
[redacted]	X	49	62
[redacted]	X	40	46
[redacted]	X	35,5	41
[redacted]	X	40	50
[redacted]	X	46	52
[redacted]	X	54	55
[redacted]	X	42	59
[redacted]	X	43	60
[redacted]	X	19,5	46
[redacted]	X	56	56
[redacted]	X	50	56
Tom	X	**47**	**51**
[redacted]	X	57	62
[redacted]	X	46	56
[redacted]	X	42,5	
[redacted]	X	52	
[redacted]	X	32	
[redacted]	X	42	
[redacted]	X	30	
[redacted]	X		61
[redacted]	X		44
[redacted]	X		49
Average		43,64	53,21

Class Y, code values (students)			
Name	**Class**	**Sum 2019**	**Sum 2021**
[redacted]	Y	34	44
[redacted]	Y	49	62,5
[redacted]	Y	47	53
[redacted]	Y	54	66
[redacted]	Y	48	68
Adam	Y	**46,5**	**42**
[redacted]	Y	47	52
[redacted]	Y	**55**	59
[redacted]	Y	51,5	56
[redacted]	Y	45	58
[redacted]	Y	36	56
[redacted]	Y	50	59
[redacted]	Y	49	60
[redacted]	Y	50	52
[redacted]	Y	41	58
Erica	Y	**32**	**58**
[redacted]	Y	44	34
[redacted]	Y	36	
[redacted]	Y	42	
[redacted]	Y	47,5	
[redacted]	Y	40	
[redacted]	Y	52	
[redacted]	Y		59
[redacted]	Y		58
Average		45,30	55,50

Table 38: Number of students in the two classes in each level of code value sum.

Class X	2019	2021
Red	7	0
Yellow	8	6
Green	6	13
>55	3	10

Class Y	2019	2021
Red	5	1
Yellow	11	2
Green	6	16
>55	0	12

Table 37 also illustrates that three out of the four focus students increased their sum of coding values, with two of them changing color category. In the following section, the four cases are analyzed in terms of the development in their beliefs. Thereby, the tendencies indicated in the analysis of the questionnaire may be investigated and elaborated further.

8.2. Case analysis: focus students

Similar to the analysis of the four students' pre-beliefs in section 5.2, each student is here analyzed as an individual case, following the structure of a general characteristic of the students' development, including his/her attitude towards mathematics, followed by an analysis of the student's beliefs about the three forms of OJ. The analyses of the students' post-beliefs are of course more comprehensive, as they include their post-beliefs, as well as a comparison with their pre-beliefs and a characterization of the development in these. In addition to data from questionnaires and interviews, the analyses now also build on classroom observations, from which situations are included as examples when relevant.

8.2.1. The case of Tom, class X

Judging from a brief view of the collected data, Tom has not experienced any noteworthy development in his attitude towards mathematics, nor in his beliefs about mathematics as a discipline. Yet, when investigating the data carefully, interesting details show there are signs of potential development and examples of situations that might contribute to such a development.

There is only a slight increase in Tom's total sum of coding values in the questionnaire, from 47 to 51, primarily because he—in contrast to 2019—does not skip any questions or answer "I don't know" in 2021. As in 2019, he makes it very clear in both questionnaire and interview that, although he mostly finds mathematics quite easy, he still does not enjoy mathematics and that he finds it boring. Approximately half of Tom's answers in the questionnaire indicate a negative emotional disposition towards the subject, as exemplified below:

4. *What do you like the most about math?* Nothing really, but I like basketball.

B2. *Why do you think somebody came up with mathematics?* Because they were bored.

In the interview, it becomes clear that this negativity or motivational state is not restricted to mathematics, but relates to school in general:

18	Tom:	It's coercion, so to speak.
19	Maria:	It's coercion?
20	Tom:	Well, it's not actually coercion to go to school, but it's 'teaching coercion'.
(…)		
23	Maria:	Do you feel that way about all the subjects in school or is it especially math?
24	Tom:	Nah, it's just… going to school is not fun.
25	Maria:	Okay?
26	Tom:	Not the subjects, anyway. (…) Being social and stuff is fun enough.

This general attitude towards school, of course, affects Tom's motivation and his attitude towards mathematics. His learning behavior is quite similar to the one he described in 2019; being confident and active in class, but with the main goal of finishing quickly so that homework can be avoided when possible.

Another important factor in Tom's motivation is the social aspect of school. He states in the interview that he learns the most from working in a group, but that the composition of the group is essential, emphasizing that the group members need to be on the same performance level and preferably be friends.

Tom's attitude towards mathematics in 2021 thus seems to be equivalent to 2019 in regard to emotional disposition, vision of mathematics, and perceived competence.

Tom's post-beliefs about the application of mathematics

There also seems to be some resemblance between Tom's beliefs about the application of mathematics in 2019 and 2021. He still finds mathematics important to learn but, similar to what he explained in 2019, this only applies to basic arithmetic, as his answer to question A1 in 2021 shows:

A1. *Do you think it is important for you to learn math?* Yes, ***because:*** you can use it, but you do not need to learn all the unnecessary stuff, only the basics.

According to Tom, more advanced mathematics should only be taught in higher education for the jobs in which it is needed, as it is not necessary for daily life. He partly justifies his view with his own experience:

263 Maria: So there is none of the math that you learn now that you think [is useful]?
264 Tom: I haven't needed it so far, anyway. Apart from adding two and two when calculating whether we can advance from the group stage [to the knockout stage in international football tournaments].

Thereby, Tom generally sees the purpose of more advanced mathematics as work-related, both from a personal perspective:

267 Maria: Do you think you will need it [mathematics] in the future?
268 Tom: That depends what my job is.

—and from a general perspective, when challenged on his view about only learning basic arithmetic:

311 Maria: [W]hy was it decided that all children in Denmark must learn mathematics?
312 Tom: Yeah… I don't know. Education. That's where it's important.
(…)
320 Tom: In upper secondary, anyway, you need to have math.
321 Maria: Yes, that's right. But is it important to learn [mathematics] in upper secondary, you think?
322 Tom: If it has a little to do with the job you will have, then it's a good idea.

In this excerpt of the interview, Tom to some extent abandons the view that mathematical content, which exceeds basic arithmetic, solely belongs in higher education, although he is consistent in relating it to work and not to general competence in life.

In contrast, Tom appears to have a more nuanced view on the application of mathematics in society. Several of his answers in the questionnaire indicate some degree of reflection about this:

A2. *Do you think it is important for everybody in Denmark to learn math?* Yes, *because:* so that you can think for yourself.

A3. *What do you experience and think that math is used for in your daily life?* If you need to build something.

A4. *What do you experience and think that math is used for in society?* Economy

Still, he is consistent in his view concerning which level of mathematics is important:

305	Maria:	Is mathematics important, do you think? In general?
306	Tom:	In connection to culture, then yes. Because math is used for everything today. But again, it's the basics – what time it is and… what's higher: four or five, in a match, right?

These indications of beliefs might actually offer an explanation for Tom's lack of motivation in mathematics, as he apparently does not see the relevance of learning most of the mathematics taught in school, as seen for example in his answer to question C3 in the questionnaire:

C3. *What is a mathematical problem?* It is too boring, and there are too many unnecessary calculations.

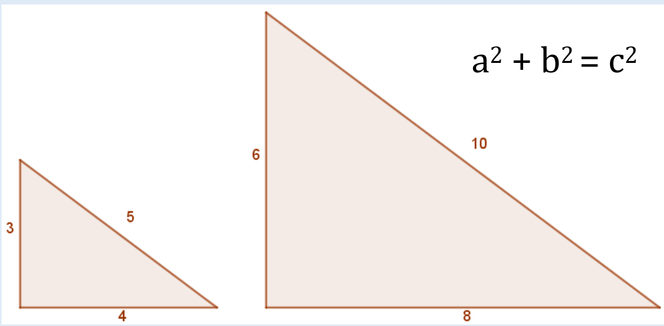
In the interview, he agrees that he is highly motivated by relevance and meaning, which of course is in opposition to the belief that (most) mathematics is irrelevant and meaningless. Also, Tom's drawing in question D1 in the questionnaire is consistent with his beliefs about mathematics being necessary and useful, but rather uninteresting, as the mathematics generally applied in reality is very basic (figure 26):

Figure 26: Tom's drawing in question D1, post-questionnaire ("Draw yourself working with mathematics that you find fun"). The explanation below the drawing reads: "Football field. The mathematics in it is not super fun, but the sport is."

Even though Tom's beliefs about the application of mathematics seem to be more or less unchanged after the two years of intervention, the level of consistency has increased, as he in all statements maintains the viewpoint that for most people, basic arithmetic is adequate to learn for coping in life. He offers more examples and a higher degree of justification than in 2019. He obviously continues to believe that some level of mathematics is important to learn in general but finds it difficult to see the relevance of learning more advanced mathematics in school. Although he provides varied examples of the application of mathematics in society, he does not seem to connect it to people's general mathematical competence nor to daily life. Thereby, Tom's beliefs still do not seem to be based on evidence.

When I ask Tom if he has learned anything in school about the application of mathematics, he answers "No, I don't think so", and he does not feel that he knows more about this issue than he did two years earlier. Nevertheless, he does at one point mention one example from the intervention; connecting Pythagoras to the job of a carpenter. This might not be coincidental, as Tom in fact also used this example several times in the short midway interview I conducted in November 2020. A few weeks before that interview, the class learned about Pythagoras and his theorem, and this topic seemed to have made an impression on Tom (example 1T).

Example 1T: In October 2020, the class was introduced to Pythagoras and his theorem. After an introduction to the life of Pythagoras, the teacher presented the Pythagorean theorem and showed two examples of right-angled triangles on the smartboard: one with side lengths 3, 4 and 5, and one with side lengths 6, 8 and 10.



The students were then asked to work in small groups, constructing right-angled triangles of their own and checking the validity of the theorem. Tom's group finished quickly, discovering that often the hypotenuse would not be an integer even though the catheti were. The teacher then suggested that the group search for the "next" Pythagorean triple following the two examples on the smartboard. This challenge seemed to motivate and engage Tom in a way that made him put aside his tendency to finish quickly. As the only one in his group, he pursued a solution for the rest of the lesson, continuing even during the teacher's wrap-up. Unfortunately, the video observation does not include his solution.

In the short midway interview a few weeks later, Tom refers to this lesson several times, including mentioning the job of a carpenter as an example of the application of mathematics:

131	Maria:	So there is nothing you are learning at the moment that you believe you can use?
132	Tom:	Well… If I become a carpenter, I can use Pythagoras.

This example was given by the teacher in the lesson to illustrate the usefulness of the theorem, and Tom repeats it in 2021. Since Tom still use it in 2021 as an exemplification of people who uses mathematics in their jobs, it seems that Tom might see the Pythagorean theorem as an example of relevant and applicable mathematics, although he still links it to work situations:

325 Maria: So people learn mathematics because of the job they are going to have. Do they use mathematics otherwise, I mean in their daily lives, in their spare time?
326 Tom: Yes, I guess so, but it’s not like you calculate Pythagoras in your spare time.

Tom’s post-beliefs about the historical development of mathematics

In the midway interview, Tom also used his knowledge about Pythagoras to exemplify the historical development of mathematics and justify his beliefs about mathematics evolving over time:

156 Tom: It [mathematics] has evolved.
157 Maria: How so?
158 Tom: It’s… well, they figured out more things. Because Pythagoras hasn’t always been there. He figured that [the theorem] out.

In the post-interview, he elaborates on his beliefs about the origin and development of mathematics, starting by perceiving it as invented:

357 Maria: […] we talked about before where mathematics originated. You wrote India and Egypt?
358 Tom: Yes.
359 Maria: Yes, say something more about that please.
360 Tom: Yes, well, they invented it, so to speak.

However, when I challenge his view and ask him to elaborate, he appears to reflect on the matter while speaking, as the following dialogue shows:

385 Maria: [Simplified: Asks if he thinks mathematics have been discovered or invented]
386 Tom: I don’t know. I guess it’s fifty-fifty.
387 Maria: Okay?
388 Tom: They figured out how to use it.
389 Maria: Yes? They discovered some and then invented some more, or…?
390 Tom: Yes. Or… I guess it [mathematics] always worked; it’s just how they figured it out.
391 Maria: Okay, then it sounds a bit like it’s discovered?
392 Tom: Yes, you can say that.
393 Maria: Yes, okay.
394 Tom: There are lots of things that we haven’t discovered yet.

Tom's comment in statement 390 can be interpreted as coherent with his reference to Pythagoras in 2020, indicating that Tom might still draw on that experience when asked about the historical development of mathematics.

There are thus indications that Tom's beliefs about the historical development of mathematics have developed during the two years of intervention. In 2019, he was hardly able to answer questions concerning the matter. However, in 2021, his thoughts seem both reflected and justified. Whether this is due to the intervention is of course difficult to know, as his general development, inputs from non-school sources, or even his mood of the day may be important reasons or factors in his level of reflection. Still, the situation described in example 1T combined with his reference to Pythagoras in 2020, and the comment in line 390, suggest that the intervention has in fact had some sort of impact on his beliefs, which to a higher degree seem to be built on evidence. What then becomes interesting is what made that particular situation special. A clarification of this matter may be found when analyzing the situation in relation to Tom's beliefs about the nature of mathematics.

Tom's post-beliefs about the nature of mathematics as a subject area

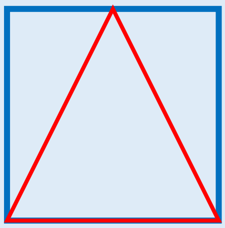
The activity in the lesson about Pythagoras offers Tom an experience to engage in mathematical processes (inquiry and systemization) to a problem and shows a rare example of Tom being motivated. He more or less 'forgets' his negative attitude and his beliefs about the (ir)relevance of mathematics, as he immerses himself in the search for the next Pythagorean triple. The previous impression of Tom being motivated by relevance thus might not only be connected to relevance in an application aspect. It seems that he perhaps also finds motivation in activities that feel meaningful and relevant within mathematics itself. Furthermore, this activity apparently challenged Tom in just the right amount, not being too easy or too difficult, and he even mentions it when asked about his preferences in mathematics in the midway interview in 2020:

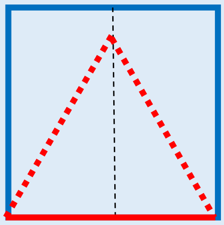
111	Maria:	If you had to choose to do some math, what would it be?
112	Tom:	I don't know. Pythagoras, maybe?

Tom's behavior in the situation was quite noticeable and uncharacteristic for him. However, the classroom observations show similar behavior in another situation (example 2T).

Example 2T: A couple of weeks after the lesson about Pythagoras, the class worked with Euclid and his Proposition 1 of Book I in *Elements* (e.g., in Heath, 1956), concerning the construction of an equilateral triangle using only a straightedge and a compass. Tom was active in both group work and classroom discussions, and he seemed engaged in the mathematical content and working methods. At one point, the students were asked to work in groups and attempt to construct an equilateral

Tom presented the idea of his group: to insert the triangle in a quadrangle, making the baseline of the quadrangle the baseline of the triangle, and then placing the third point in the middle of the opposite side of the quadrangle. The teacher made Tom and the rest of class aware that this triangle was an isosceles, not an equilateral triangle.





Tom then had a new idea: "I just thought of something," he exclaimed, and then used the perpendicular bisector of the baseline to help him find the third point by measuring when the distance between the end of the baseline and the intersection with the bisector equaled the length of the baseline. A strategy that almost resembled the use of a compass.

The teacher acknowledged his idea, but unfortunately missed the opportunity to relate it to the construction in Euclid's proposition.

Firstly, this example shows that Tom is capable of thinking mathematically and coming up with an alternative strategy when the first one turns out not to be successful. Secondly, it indicates that Tom finds motivation in this form of inquiry activity. Perhaps this motivation is initiated by the challenge he received while working with Pythagorean triples. Judging from these two situations, Tom appears to enjoy an inquiry-based approach that requires mathematical reasoning, and, as mentioned, it seems that they have had a small yet long-lasting influence on his beliefs.

Nevertheless, this influence is not apparent when it comes to Tom's overall perspective on mathematics. About a month after example 2T, the students were once again sent home to attend online teaching, which ended up lasting five months. During these months, the teaching did not include the same type of opportunities for working with mathematical methods and investigation, which partially had to do with the teacher's lack of experience and willingness to try new things in an online environment with which she did not feel comfortable. In addition, the class as a whole struggled with motivation and participation, and the teacher felt obliged to prioritize relational and pedagogical work. As for many students, this period was difficult for Tom, especially in a social aspect. He participated in the teaching and was somewhat active, but it is quite clear from the observations that he was missing his classmates. Most of the time during group work was spent on small talk and joking around. The class worked with two projects during this period (both of them in groups). Tom worked dedicatedly and seemed to enjoy this form of activity, but the level of mathematical content was quite low. He still appeared to be focused on finishing as quickly and easily as possible, and by the end of the intervention, his statements in the interview still indicate an overall dualistic perspective on mathematics. For example, he sees himself as a good mathematics student because of speed and accuracy:

152	Tom:	I finish faster than others, my tasks are correctly solved, and I help the other students.

Even when I asked him, during the short midway interview, about the situation described in example 1T, Tom seemed to reduce his engagement to a matter of competition, which is connected to a focus on speed and correctness:

34	Tom:	Well, I needed to find it [the next Pythagorean triple]. Because I had to find it before the others did.
35	Maria:	Oh, I see. […] Sometimes […] you like to finish quickly, right? […] But you didn't this time, you actually persisted longer than you had to.
36	Tom:	That's because I have to do something. I cannot just sit there and do nothing. […] I have to finish, so I don't have to do it at home.

Instead of perceiving this experience as a situation of motivation and engagement, Tom thus seemed to accommodate it to his previously described beliefs about himself as indifferent, competitive, and eager to finish. He excused his perseverance as passing time even though he actually persisted working beyond the acquired time.

Summary

Hence, this is indeed an illustration of the complexity of changing students' beliefs, particularly concerning the time perspective. Even after two years of intervention, Tom's beliefs have not changed notably, still being rather limited. However, there are signs of development. The described situation regarding the Pythagorean theorem has assumingly provided Tom with experiences and concrete examples concerning all three forms of overview and judgment. Although these experiences and the knowledge that Tom has obtained along with them do not seem to have changed his beliefs notably, they may have contributed to making his beliefs more reflected in terms of consistency, exemplification, and justification. What is further noticeable is the possible factors that might influence the students' outcome of this intervention. In the case of Tom, factors such as motivation, social dynamics, other dimensions in the belief system (e.g., about mathematics education or self, relation to the teacher, etc.) may all potentially have affected how Tom responds to the intervention in terms of development in his beliefs.

For example, it might be that his beliefs about himself as a learner of mathematics (in short: talented and competitive but indifferent) are so central and psychologically important to Tom that a few situations providing contrasting experiences have not been sufficient to change this. Thereby, these experiences were rejected or perhaps modified to fit his existing beliefs (cf. section 2.4). Furthermore, the teaching during the periods of lockdown did not, to the same degree, include situations requiring reasoning, inquiry, etc., and Tom did therefore not experience the same number of opportunities for developing his beliefs.

The case of Tom thereby suggests that for some students, a development in beliefs requires a continuous and vast number of situations, carefully designed to provide evidence for more evidentially based beliefs – with considerations of possible influencing factors included in this design.

8.2.2. The case of Molly, class X

As in the case of Tom, Molly's sum of answers in the questionnaire has only increased slightly, from 57 to 61. She thereby remains in the green category, which can largely be attributed to her elaborate answers. In 2019, Molly expressed enjoyment of mathematics and a high perception of her own performance level. Now, she states that she finds mathematics "okay", and she rates her own performance level at the slightly unhappy emoticon in the questionnaire, explaining in the interview that she finds mathematics a bit more difficult.

Molly still seems to associate mathematics with its content areas, especially arithmetic:

> *1.* *What do you think of when you hear the word "mathematics"?* Addition, subtraction, multiplication, fraction.

This is consistent throughout the questionnaire and the interview. Like in 2019, her drawing shows her sitting at a desk, this time doing mathematical problem solving on a computer, without any emotional expression though (figure 27).

She emphasizes her enjoyment of "templates", basic skills exercises, and mastering methods. However, she also appears to appreciate the orientation towards application in the mathematics lessons:

> *C1.* *What do you think the difference is between mathematics and other subjects?* There is not a big difference, but I suppose I like the other subjects better. But in math we learn about things like taxes etc.

The classroom observations show Molly as a hardworking and rather shy student who does not participate much in class discussions. In the interview, she mentions that the learning environment in the classroom continues to contribute to a feeling of insecurity:

67 Maria: How about in class, when you have a classroom discussion, do you participate and raise your hand?

68 Molly: If I can answer, but I am not completely sure, I fall back a bit.

69 Maria: Okay?

70 Molly: Because I know… because you feel a little afraid that your answer is wrong.

(…)

82 Molly: Maybe, if they [the other students] give you sarcastic remarks: "Why did you say that, that is not even close?" […] You are afraid to say it.

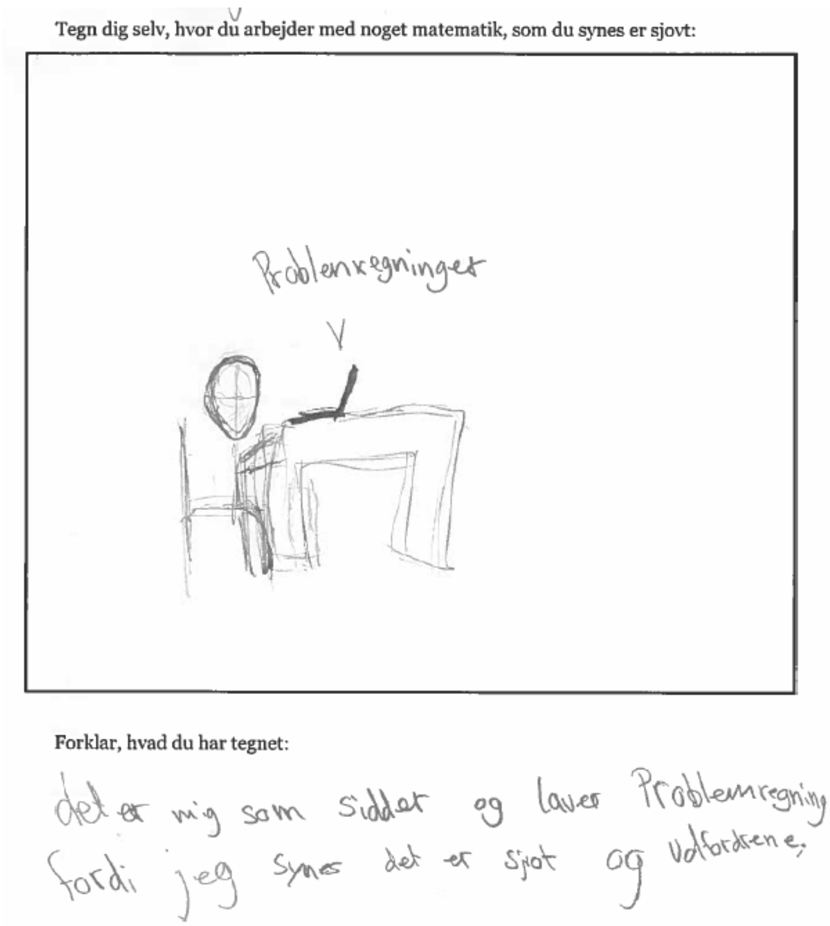


Figure 27: Molly's drawing in question D1, post-questionnaire ("Draw yourself working with mathematics that you find fun"). The explanation below the drawing reads: "It is me sitting and doing problem solving because I find it fun and challenging".

Overall, it thus seems that Molly's attitude has changed a bit with a slightly more negative emotional disposition and a decrease in her perceived competence and her self-confidence, perhaps because of the class environment.

Molly's post-beliefs about the application of mathematics

In 2019, Molly's beliefs about the application of mathematics seemed to have a high degree of exemplification, justification, and consistency. This has not changed. In the questionnaire as well as in the interview, she emphasizes repeatedly that it is important to learn mathematics, as "it is used every day", and "in our lives, there is always something that is mathematics". In the questionnaire, she uses everyday examples of the application of mathematics, even in a societal context:

A4. What do you experience and think that mathematics is used for in society? When you shop, drive, work, cook, and much more, because it is used every day.

In the interview, she is however able to distinguish further between the use of mathematics in daily life and in society. Here, she describes how she personally experiences using mathematics for school, for doubling recipes when she is baking, for shopping, and in relation to foreign currency. Molly also imagines that she will need mathematics in her dream job as a fashion designer:

173	Maria	So that might be the education you want? Do you think that there is mathematics in that?
174	Molly:	I don't know. Well, yes, how much you sell in a year, and statistics of how many customers you have in a week or a month.

Her examples are thus related to counting and multiplying. This rather narrow perception of mathematics is repeated when I ask her what would happen if people did not learn mathematics:

194	Molly:	[…] then it would be difficult to … Yes, then you would have to count everything instead of using the multiplication tables, because every time I count, I just use the two-table or the five-table.

Yet, she also exemplifies the application of mathematics with her father using mathematics when paying taxes, and when asked about the use of mathematics in society, she mentions energy and water consumption.

Hence, Molly's beliefs about the application of mathematics still seem reflected and evidentially held, but there are practically no signs of development or connection to the intervention.

Molly's post-beliefs about the historical development of mathematics

In relation to the historical development of mathematics, Molly seems to have given it more thought now than in 2019, when she was hesitant in her answers, and her beliefs were inconsistent.

Now, in the questionnaire and in the interview, Molly is consistent in her statements concerning the origin of mathematics. In both situations, she states that it comes from Egypt, and that one person discovered it when starting to add numbers. She seems more conscious of her beliefs, for example, when asked if mathematics has changed since it was invented—something that she did not really have an answer for in 2019. Also, Molly's level of justification seems higher, although she does not provide concrete examples:

265	Maria:	Is it the same mathematics [that you learn in school now], or has it changed?
266	Molly:	I think it might have changed a bit.
267	Maria:	Yes? How?
268	Molly:	Because maybe it is explained differently now
269	Maria:	[…] But is it the same, in reality? I mean, is it just the explanations that have changed?
270	Molly:	I guess. The explanations and the methods.

However, as in the case of Tom, the lessons about Pythagoras (example 1T, section 8.2.1) seem to have made an impression on Molly, both in relation to the mathematical and the historical content. In the midway interview, I asked her if she felt that she had gained more knowledge about the history of mathematics during the intervention:

92	Molly:	Well, we have just heard about that guy, Pythagoras. (…) I think that it's quite cool that he invented that "*a* squared and *b* squared equals *c* squared".

Now, at the end of the intervention, it is still present in Molly's memory, as it is included in several of her answers in the questionnaire:

B4.	*What did you learn in school about the history of mathematics?* Pythagoras
E1.	*What do you remember best from the mathematics teaching in 6th and 7th grade?* Pythagoras, fractions and multiplication
E2.	*When do you think you have learned the most and why?* Pythagoras

Hence, Molly is, here, able to exemplify her experiences with the historical development of mathematics. In the excerpt from the midway interview above, she even mentions that Pythagoras *invented* his theorem—despite her belief that mathematics is discovered. Perhaps a potential cognitive conflict awaits here, if she is given the chance to reflect on the matter.

Compared to the beginning of the intervention, Molly's beliefs about the historical development of mathematics appear more conscious with an increased level of consistency, justification, and to some extent, exemplification. Yet, with only one concrete example, it seems premature to label them as evidentially held.

Molly's post-beliefs about the nature of mathematics as a subject area

An equivalent change cannot be detected in Molly's beliefs about the nature of mathematics as a subject area. As the reader may recall, these beliefs were (in 2019) of a dualistic character and largely related to school mathematics, emphasizing arithmetic, rules, skills and correctness in both results and methods. This also seems to be the case in 2021, and it is prominent in both questionnaire and interview. For example, she describes a mathematical problem (question C3) as "an arithmetic skills exercise [with no use of calculator]", and in question C2 concerning what she finds the most important in math, she chooses the options "b. To know rules and formulas by heart", "c. To use the correct methods", and "f. To be able to explain what you think", which can all be perceived as success criteria in 'traditional' school mathematics.

Molly mentions "understanding mathematics" several times as an important goal for her, but it appears that she merely perceives understanding as a prerequisite for solving tasks correctly. For example, she states that she prefers to work with peers who are on a slightly higher performance level, because then they may explain to her "how they found the result, so that I know it the next time." In addition, Molly describes a high performing student as someone who can help the others, as well as solve the tasks faster, thus indicating that she associates mathematics with a competitive element.

In the interview, I ask her what mathematics is "about":

277	Maria:	What is mathematics about, to you?
278	Molly:	Mostly addition and subtraction, and to… yes, to learn it. […] And the multiplication tables too.

With the term "learn", Molly clearly refers to the school subject of mathematics here.

Depending on the receiver, these questions may, of course, be perceived as in fact dealing with the school subject of mathematics. Yet, when Molly is asked about mathematics that is clearly not in a school context, her beliefs about mathematics still seem to be restricted to numbers and calculations:

C4. *What do you think a mathematician does?* To be honest, I don't know, but I think they figure out new methods to do calculations.

313	Maria:	[…] Several times, we have talked about what a mathematician might do. Have you become any wiser on that matter?
314	Molly:	Yeah… finds new ways to calculate things, maybe?

Molly thereby does not seem to have changed her beliefs about the nature of mathematics, and the potential for development towards a more process-oriented approach to the subject, that could be detected in 2019, has not been realized. Nevertheless, Molly's beliefs about this aspect are consistent, justified, and exemplified, as she probably has plenty of experiences from the mathematics classroom to confirm these beliefs. Furthermore, the classroom observations show how the element of reflection is often omitted or forgotten, as in the following example:

Example 1M: In the lesson described in section 7.4.1 concerning probability and the distribution problem of Pascal and Fermat, the teacher asked the students to play the game in pairs, stop when the score is 2–1, and discuss a fair distribution of the stakes. After the first game, Molly suggested a distribution of 15–5 in favor of the player with 2 points. Her reasoning behind this suggestion was that the player with more points should have more money. Her partner agreed. In the subsequent classroom discussion, the teacher asked all the pairs to present and argue for their suggestions. Six out of eight group suggested the 15–5 solution, one pair suggested that the players get their €10 stakes back, and the last pair recommended a distribution of 13.3–16.6, explaining that they divided the €20 with the three points, and thus made the distribution accordingly. However, none of the groups proposing 15–5 was asked to justify their choice of amount. One group even mentioned the corresponding percentages, but the students were not asked to explain the *mathematical* reasoning behind their suggestions, nor to reflect on the difference between the presented solutions. Instead, the teacher simply acknowledged the proposals and moved on to present Fermat's solution.

Summary

The overall interpretation of Molly's beliefs about mathematics as a discipline is thus that the beliefs connected to the application and the nature of mathematics are more or less unchanged, but that her beliefs about the historical development have become more reflected. This pattern may likely be related to the centrality of her beliefs. In 2019, her beliefs about the historical development were, if not non-existent, then very peripheral. In contrast, her beliefs about the two other aspects seemed to be quite central and based on evidence, especially on her experiences in the classroom. Within each aspect or belief cluster, Molly's beliefs are in fact exemplified, justified and consistent, but there lies a large potential for cognitive conflict between contradictory beliefs, which could lead to a higher degree of reflection and nuance. For example, her beliefs about mathematics being restricted to arithmetic calculations does not match all her examples of the application of mathematics, nor do her beliefs about mathematics being discovered match her statements about Pythagoras inventing his theorem.

8.2.3. The case of Erica, class Y

When we first met Erica she was in the beginning of a development towards a more positive attitude towards mathematics, as well as an increased self-confidence as a mathematics learner. As described in section 5.2.3, Erica previously struggled with mathematics.

In June 2021, it seems that she continued her positive development during the two years of intervention, which is particularly clear from her response to the second questionnaire. Here, Erica gives thorough and elaborate answers, which—as opposed to her answers in 2019—generally indicate a positive attitude towards mathematics. The total sum of the coding values assigned to her answers has increased from 32 to 57, placing her in the high end of the green category. This increase is the largest in her class and the second largest of all the students. A very illustrative example of this development is her answer to question 1:

1. *What do you think of when you hear the word "mathematics"?* I think of addition, subtraction, multiplication, division, fractions. And that we are going to have fun with math, and not think about if what you are doing is wrong, and that no one will laugh at you if you make the *cool mistake of the day*[9]. But also that you [...] actually feel like going to class with a positive attitude.

Erica marks the happiest emoticon for her enjoyment of the subject (question 2) and the slightly happy for her level of performance (question 3). This is a drastic change from 2019, when she marked the unhappiest emoticon in question 2 and the slightly unhappy in question 3. It is clear that she finds the learning environment safe and suitable for learning:

4. *What do you like the most about math?* That you always learn something new and that you have the opportunity to get the help you need, and that there is no stress about whether what you do is correct or wrong, and that you have the time to learn.

However, during the post-interview, it is difficult to detect the positivity found in the questionnaire. Erica appeared tired and in a bad mood when I met her for the interview only four days after completing the questionnaire. She hardly answered my questions, and then mostly with a shrug, a single word, or "I don't know". Only on a few occasions, when asked directly about her responses to the questionnaire, did she reveal a hint of positive attitude (e.g., finding it fun to work on projects).

A subsequent conversation with her teacher revealed that Erica's apathy probably related to an episode that happened just before the interview; Erica was not permitted to work with her friend. Hence, she was rather dissatisfied with her teacher when I met her for the interview, and she was not set to be interviewed at that particular time. Moreover, Erica mentions during the interview that she generally dislikes school at the moment. She finds most subjects boring (including mathematics) and she struggles

[9] The teacher of class Y introduced the concept of "cool mistake of the day" early in the intervention, partly to illustrate how mistakes can be basis for learning, and partly to create a safe learning environment in the classroom. In every lesson, the teacher complimented a student who had made a mistake on which learning could be build.

with her concentration—something that she relates to the periods with online teaching caused by the COVID-19 pandemic, which she found extremely difficult.

This situation is a clear example of difficulties connected to qualitative research, especially with children as subjects. One cannot expect an individual always to be prepared and willing to enter into the role of a subject of investigation. People (and maybe teenagers in particular) have feelings, mood swings, good days, and bad days that all affect their participation, their attitude, and their willingness to share their thoughts. But does this mean that her statements in the interview are not valid? Or does it perhaps give us an idea of the complexity in the affective field of mathematics education research in general and of the concept of beliefs in particular, cf. the considerations dealt with in chapter 2 concerning accessing and assessing beliefs?

The quality of the data from this interview must thus be considered: Is it useable? Does it merely reflect Erica's mood, or is it possible to infer her beliefs? Which data reflect her beliefs and attitude most accurately—the interview or the questionnaire? A possible solution may be found in the data from the classroom observations, which might provide the nuances needed to assess Erica's beliefs.

Looking through the classroom observations, examples of both "sides" of Erica's development appear. Most of them clearly show how Erica's attitude towards mathematics has changed, supporting the data from the questionnaire, but during the second period of online teaching, her motivation clearly decreased. She seemed tired and was sometimes lying in bed during the lessons. She rarely participated actively during the lessons, and the teacher told me at one point that she was worried about Erica and thus had suggested that Erica attend the 'emergency' teaching at the school. Here, students who were in need of support, for example, because their parents were not able to stay home from work, or because they did not thrive in the solitude of online teaching, could come and attend the online teaching in small groups with the presence of a teacher. In the interview, Erica describes this period as "difficult", making it "hard to concentrate".

Apart from this period, Erica generally pays attention during class, take notes, engages in group work and participates in classroom discussions, as illustrated in the following examples which both took place within the last months of the intervention:

Example 1E: In a project called "With the help of mathematics" that took place in March 2021 during the second period of online teaching, the students were planning their dream vacation. Erica was working with another girl, and the pair had experienced some difficulties getting started with the project, receiving quite a lot of support from the teacher to deal with this open task. As part of the project, the students were asked to investigate how mathematics could help them make decisions related to their luggage (e.g., concerning volume, weight, etc.). I participated briefly in the online meeting room of Erica's group while they worked on this task. The two girls were very engaged in the problem and had already calculated the volume of the suitcase that they planned to use. As I entered the room, they were vividly discussing what they might put into the suitcase. Erica was keen on bringing her duvet, which led to a discussion of how much space a duvet would occupy in the suitcase and how they could make room for clothes, etc. With a little support from me, Erica got the idea to place the clothes on top of the duvet and roll it all tightly to get the air out of the duvet. I let the group continue on their own. The next day, the groups gave a short presentation of their "luggage math" to the rest of the class. Clearly, Erica's group continued their investigations the day before, and had—in addition to working with the volume of the suitcase and its contents—also investigated the weight of their luggage and compared it to the restrictions from the airline. Considering their struggles at the beginning of the project, they had thus managed to work independently and inquiry-based with a low level of support.

Example 2E: In the lesson about the sum of angles in a triangle described in section 7.4.3, Erica participated actively in both group work and classroom discussions. She was engaged in the tasks and took the lead in her group. She gave input concerning all issues raised in the classroom discussion even though she was not always sure of the correctness, and she provided several arguments, for example, after the students had placed cut-off corners on a straight line: "You cannot say anything with certainty yet. Maybe the drawn lines are crooked, or you have measured wrongly." Erica thus engaged in mathematical endeavors, made mathematical arguments, and showed both confidence and motivation.

These examples make a remarkable contrast to the resistance and low level of confidence that Erica showed in the beginning of the intervention.

To sum up, Erica might sometimes have struggled with her motivation and her general attitude towards school, but she also seems to have developed a more positive attitude towards mathematics in all three dimensions: her emotional disposition, her vision of mathematics, and her perceived competence. These two rather different and maybe even contradictory perspectives seem to be present at the same time. As this study seeks to investigate a possible development in the students' *beliefs*, and not motivation, I will in the following sections present data that indicate a development in beliefs, in terms of exemplification, justification, and consistency. Hence, I have chosen to interpret Erica's response to the questionnaire as more representative of the development in her beliefs than her statements in the interview, supported by the overall impression from the classroom observations. This does not mean that Erica's statements in the interview are not valid or true. However, it is likely that they mainly reflect

Erica's general struggles with school, friends, authorities, etc. and, in the analysis of the interview, I thus attempt to focus on issues that are primarily related to mathematics.

Erica's post-beliefs about the application of mathematics

Most of Erica's answers to questions A1-A5 in the questionnaire do not indicate any noteworthy change in Erica's beliefs about the application of mathematics. She still finds mathematics important in general ("mathematics is one of the most important things in our society"), but is initially not able to give a reason for this. Personally, she perceives mathematics as important for exams and education, but she still does not experience using mathematics in her daily life apart from shopping-related situations:

> *A3.* *What do you experience and think that math is used for in your daily life?* I don't use it as such in my daily life, but I use it to figure out if I have enough money for what I want to buy, or how much money I have left when I have bought something.

In the interview, she does not give any examples of the application of mathematics in society or in her personal life or the lives of others. Still, she confirms that she finds it necessary to learn mathematics for educational purposes. Thereby, Erica does not appear to have changed her beliefs about the relevance of mathematics for daily life. However, when it comes to the application of mathematics in society, there are indications of change in her beliefs. In 2019, Erica was not able to exemplify the application of mathematics in society, but now—in addition to shopping—her answer includes an example that is most likely related to the media's coverage of the COVID-19 pandemic. This has been a topic in some of the mathematics lessons, and Erica thus seems to draw on these experiences:

> *A4.* *What do you experience and think that math is used for in society?* I think it is used to find out how many people are sick or the percentages for how and why. You also use it to figure out how much money you save when you buy something on sale.

Interestingly, Erica provides more examples of the use of mathematics in question C4, when asked about the job of a mathematician:

> *C4.* *What do you think a mathematician does?* Calculates the big things that are happening around the world, e.g. find out how many people have coronavirus and how the weather will be.

Erica thus seems to have developed her beliefs about especially the societal application of mathematics, which to some extent appear to be evidentially held. Still, the dualistic perspective on mathematics appears to be very central to her beliefs, associating the use of mathematics primarily with money, as seen in the following example from the classroom in March 2021:

Example 3E: At the end of the above-mentioned project "With the help from mathematics", the students presented their travel plans to the rest of the class. The teaching was online, and the students had made PowerPoint slides following a template from the teacher. The presentation had to include a budget, as well as calculations connected to currency, scale ratio, and luggage. During the project, the teacher emphasized several times that the main focus of the project was on how mathematics can contribute to decision-making and the importance of showing which calculations these decisions are based on. The presentation by Erica's group did not include any explicit calculations but instead listed a lot of numbers, which in most cases (except the mandatory elements related to the luggage and the scale ratio) were connected to money. Several of the slides did not include mathematics at all. The group completed their presentation by stating what working on this project had meant to them:

"That you can use mathematics for anything. We have planned an entire trip with the help of mathematics, which we used to find out what it will cost, and what we should bring, and how long it will take to get there. Yes, we got a lot out of working with this task."

With the last slide, Erica's group indicated that they had a strong feeling of having applied mathematics and used it to make decisions. This is, of course, partly true. However, the mathematics that they used was restricted to simple arithmetic, and they had almost no explanations of how it had helped them in their decision-making. However, two months later, classroom observations show that Erica seemed to draw on this first experience of working project-oriented (example 4E):

Example 4E: In May 2021, the class once again worked on small projects, this time about planning a party (cf. section 7.4.4.). Where her presentation of the traveling project seemed a little unsteady, Erica presented this project clearly and confidently. Although explicit calculations were still missing to some extent, the group had clearly used mathematics for several decisions, as well as in their arguments for e.g., choice of menu and placement of tables. Erica thus appeared to draw on her previous experience with doing projects in mathematics.

In summary, Erica's expressed beliefs about the application of mathematics in daily life, overall, remain unchanged and continue to revolve around the educational purpose of learning mathematics, and around shopping as the main area of application. Nevertheless, there are indications that her beliefs about the societal use of mathematics are developing, as she is able to provide more examples, and to apply mathematics in certain situations.

Erica's post-beliefs about the historical development of mathematics

In 2019, Erica did not show much sign of having ever considered issues regarding the historical development of mathematics, answering "I don't know" to most questions concerning this matter. In

2021, her answers are slightly wordier, but there are very few indications of any development in her beliefs, as the answers both in the questionnaire and in the interview are rather vague, hesitant, and sometimes even contradictory.

Erica states in the questionnaire that mathematics came into being in 1924. When asked about this in the interview, it turns out that Erica simply wrote this specific year as a representation of "a long time ago". Thereby it seems that she still has a rather simplistic historical awareness, which is confirmed when she is asked whether she thinks mathematics is more important now than 100 years ago. In the questionnaire, she answers yes, but does not give a reason for this other than: "I don't know, I just think so". During the interview, she expands this answer, stating, "At that time, people did not know what it [mathematics] was".

Like in 2019, she recalls that the Egyptians might have played a role in the development of mathematics, and she hints that she believes mathematics to be invented:

247	Maria:	Where do you think mathematics comes from?
248	Erica:	Somewhere in Egypt.
249	Maria:	Yes? How did it come into being, you think?
250	Erica:	I don't know. Someone said two plus two.

She does however believe that mathematics has changed, yet not in its essence:

270	Erica:	It has changed because it is more on the computer now.
271	Maria:	Yes, okay. Did that change mathematics?
272	Erica:	Yes. The setup.
(…)		
275	Maria:	Okay. So it looks different, or what do you think?
276	Erica:	Hm. Yes.

Erica thus seems to associate mathematics with the way numbers are organized in an algorithm during calculations, and apparently, she imagines that a change in this organization constitutes a central part of the development of mathematics.

When asked what she has learned about the history of mathematics (question B4 in the questionnaire), she answers "just that there have been many forms and people who have done something". Although this answer is very vague, it does indicate that Erica has in fact had certain experiences with the history of mathematics, even though she is unable to specify them.

Overall, Erica thereby shows a slight development in her awareness of the historical development of mathematics, even though it is rather insignificant. Although the intervention offered several opportunities for experience and examples, they have not essentially influenced Erica's beliefs, supposedly due to a low level of reflection.

Erica's post-beliefs about the nature of mathematics as a subject area

The clearest signs of development in Erica's beliefs are found in relation to the aspect of the nature of mathematics. As described in section 5.2.3, her perspective on mathematics belonged in the dualistic end of the spectrum in 2019, and she did not seem to distinguish between school mathematics and mathematics as a discipline. Two years later, her beliefs about *school* mathematics seem to be almost unchanged. When asked about mathematics in a school setting during the interview, Erica is consistent in associating the subject with numbers and calculations:

280	Maria:	What is mathematics about?
281	Erica:	Numbers.
282	Maria:	Numbers. And what do you do with these numbers?
283	Erica:	Calculate them.

She also emphasizes the importance of speed and correctness:

139	Maria:	In sixth grade, you didn't feel that you were very good at it [mathematics]. But there has been some progress? How can you sense that you have become better?
140	Erica:	I don't know.
141	Maria:	What is it that you can do now, or how can you feel an improvement?
142	Erica:	I'm faster at solving them [the tasks].

She mentions one particular student who she finds very skilled in math. When I ask her, what it is that this student is able to do, Erica struggles with an exemplification, but ends up answering:

348	Erica:	You cannot [give examples]. Everything! She is good at it all, addition, subtraction, multiplication, everything.

This thus points to a relatively narrow view of school mathematics and its success criteria. Later, this is supported by Erica's idea of what it takes to become good at math:

353	Maria:	If you wish to become better at math, what can you do then?
354	Erica:	Ask [the above-mentioned 'good' student].
355	Maria:	[…] Can you do anything yourself, you think?

356 Erica: Ask the teacher.

357 Maria: Yes?

358 Erica: Pay attention.

(…)

364 Erica: Yes, a brain that can remember it all.

These statements confirm what was seen in the lesson about Pythagoras described above, namely that Erica seems to perceive mathematics as something that is transferred from the teacher to the student. She apparently also believes that memorizing is part of learning mathematics. These are both views that belong to the dualistic end of the spectrum.

These beliefs do not appear to be consistent though. As in 2019, there seems to be exceptions from Erica's dualistic perspective on school mathematics. Erica marks the relativistic options "to be able to explain what you think", and "to get good ideas", in question C2 ("What is the most important in math?"), along with the more dualistic "to know rules and formulas by heart". However, we cannot be certain what Erica means, when she chooses these options. In fact, they could all three be interpreted in a dualistic way. It is thus difficult to categorize her choices within the spectrum. Yet, there are other examples of a relativistic view on mathematics. For example, Erica mentions persistence as another essential ability when doing mathematics:

301 Maria: You say that you are good at math when you try, right?

302 Erica: Mm [confirms].

303 Maria: How about if you try and try, but don't succeed. Are you still good at math then?

304 Erica: Yes.

There is thereby a lack of both consistency and exemplification in Erica's statements, and she does not even attempt to justify her beliefs about school mathematics.

However, Erica's answers connected to the *discipline* of mathematics in the questionnaire suggest that Erica has developed an awareness that may contribute to a more nuanced image of the subject area:

C3. *What is a mathematical problem?* It is a problem to be solved. A mathematician can solve that problem and a calculator. It can also be a mathematical problem if you have to figure something out that is hidden behind something that you have to solve to get the correct answer.

Erica's behavior in example 2E about the sum of angles in a triangle (described above) supports that she has become more conscious of the nature of mathematics as a subject area, as she engages in mathematical thinking and working methods. The more nuanced image of mathematics is also found in

Erica's drawing which depicts herself from behind giving three quite diverse examples of the mathematics that she has found fun (figure 28):

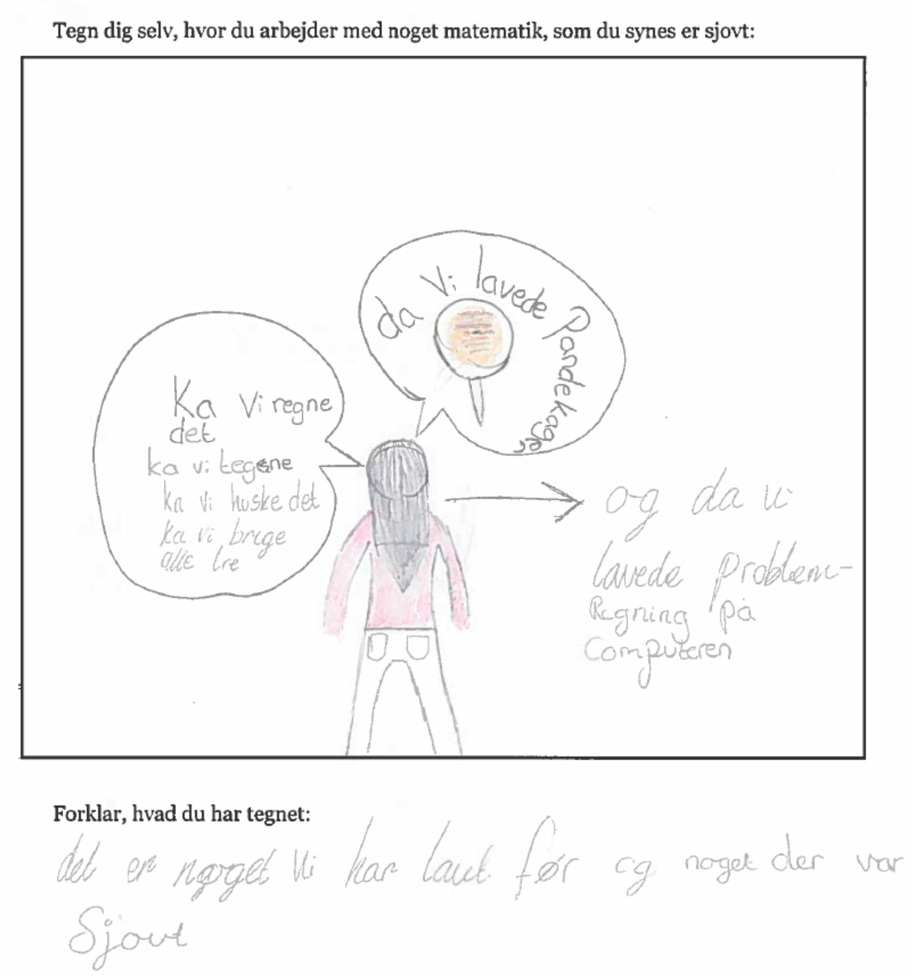


Figure 28: Erica's drawing in question D1, post-questionnaire ("Draw yourself working with mathematics that you find fun"). The text in the left speech bubble is "If we can calculate it, we can draw it, we can remember it, we can use all three". This is a rewrite of a phrase that the teacher often uses: "if we can draw it, we can calculate it". The text in the right speech bubble is "when we made pancakes", and the text on the right side of the drawing is "and when we did problem solving on the computer". Erica's explanation below the drawing reads, "It is something we have done before and something that was fun".

The first example, where Erica misquotes a sort of 'mantra' used by the teacher to support the students' problem-solving strategies, indicates that Erica repeats her recollection of it without reflecting on its meaning or purpose. In Erica's version, the phrase does not make sense. She even makes an addition to the phrase where the aspect of memorization is included, which supports the impression of Erica having a dualistic perspective on mathematics.

The second example relates to the pancake activity (cf. section 7.4.2). Erica was active in the practical part of the activity, but she did not participate in any discussion, nor did she have time to do the mathematical investigations. The reason that she has included this activity in her drawing might be that it represents a different form of teaching and a much-needed change in the routine of online learning. Although she did not complete the task, this might indicate that Erica has been offered an experience with the application of mathematics, thus making her conscious of the role of mathematics outside a school context.

The third element in Erica's drawing is not directly connected to the intervention, as it concerns an unspecified activity of problem solving on the computer, which potentially could take place in any 7th-grade mathematics lesson in Denmark. However, this element represents a third form of mathematical activity and hence, shows that Erica's perspective on mathematics includes several aspects.

Summary

In general, Erica's post-beliefs about mathematics as a discipline thus have not changed in essence, but in all three aspects of mathematical overview and judgment, there are indications of development. Although she might not be conscious of this development, she seems to provide more examples, and she indirectly draws on experiences from the intervention. Her beliefs about the nature of mathematics in a non-school setting appear to be more nuanced, but the clearest difference between 2019 and 2021 is in Erica's attitude towards mathematics. Her approach to challenges, her self-efficacy and her engagement in mathematical activities have gone from insecure and negative, to confident and positive. There are some indications that this change might be related to the intervention, but other factors can easily be involved, for example, her relationship with the teacher, her general development as an individual, or the influence of her parents.

8.2.4. The case of Adam, class Y

Adam is one out of only two students whose total coding sum has *decreased.* In 2019, his sum was 46.5, now it is 42. Similar to 2019, his answers in the post-questionnaire are short and somewhat vague, but this time, more of them have emotionally negative elements, related to his emotional disposition as well as his perceived competence. For example, he answers "boring" to question A1 ("What do you think of when you hear the word 'mathematics'?"), and he marks the slightly unhappy emoticon when asked to rate not only his enjoyment of the subject (question A2), but also his performance level (question 3). These two dimensions of his attitude seem to be connected in the sense that his emotional disposition is dependent on his perceived competence:

4. *What do you like the most about math?* When I can figure it out
5. *What do you like the least about math?* I don't understand a thing

When asked about the difference between mathematics and other subjects, he answers that he can actually keep up in other subjects. However, Adam states in the interview that he is generally struggling with his motivation in school, not just in math. This is further supported by his behavior during the interview, as he generally appears almost apathetic, answering very briefly and uninterested. He does not clearly display the negative emotions that appeared in the questionnaire, but he characterizes his feelings about mathematics as "medium" and does not especially prefer any specific topics or activities. Furthermore, he states that although he is not particularly active in class, he is not nervous about raising his hand.

Apparently, Adam's attitude does not seem to have changed much during the intervention, and he does not seem to have continued the positive development in his learning behavior that was emerging in 2019. However, the data from the midway interview conducted in November 2020 challenge this interpretation. During that interview, Adam seemed positive, confident, and motivated. He expressed enjoyment of the mathematics lessons, feeling that they had become "more fun". Adam also described how he had changed his learning behavior, working more concentrated and participating more actively in class, relating this change to his increased motivation:

46	Adam:	I don't think it is the teaching as such that has changed; I just think it is because I started paying more attention [in class].
47	Maria:	Okay. So it also changes how you feel about it [mathematics]?
48	Adam:	Yes.

Experiences of success seemed to have contributed to his motivation:

13	Maria:	If you think about what you have been doing [in class] since the summer holiday, what did you like best?
14	Adam:	Um, I don't know … when we did multiplication with decimal numbers.
15	Maria:	Yes? What did you like about that?
16	Adam:	It was just fun, when I figured out how to do it.

Classroom observations from the end of 2020 support this narrative. Here, he seemed to work in a concentrated manner and often asked the teacher for help if he did not understand the task or struggled with a certain method. Although he was still rather quiet in classroom discussions, he did sometimes contribute, and he mostly appeared to pay attention.

Unfortunately, it seems that this positive development in Adam's motivation somehow stopped along the way. Field notes from the online teaching confirm what Adam already indicated in the midway interview, namely that the periods of lockdown were extremely difficult for him. In the following quote from the midway interview, he describes the first lockdown:

36 Adam: Well, it was just difficult to keep track of all the homework and things like that. And then my mom got mad at me, because there was a lot of things I couldn't figure out. There were many things that I didn't know where to look for or anything. It was just hard. And then my mom got upset with me very often.

The data from 2021 indicate that the second period of lockdown, which started two months after the midway interview, was even more difficult for Adam and indeed affected his enthusiasm. Field notes from this period show that he was truly struggling with his motivation and general well-being. He very often had his camera turned off or pointed it towards the ceiling. He was late for many of the lessons; he was often lying in bed, and he only participated when the teacher specifically asked him to. During whole-class teaching as well as group work, he was very passive and often let others do the work. In more than one debriefing meeting with the teacher, she expressed her concern for him and described how she was in contact with his parents. In the following example, Adam expresses his struggles:

Example 1A: As described in example 1E (section 8.2.3), the class worked on a project about planning a vacation in March 2021. Adam worked with a student with whom he normally collaborates. The few times I joined their online meetings during the preparation phase, they hardly spoke, and I got the impression that they were working individually—or rather that Adam's partner was working individually. When the partner asked questions or tried to start a discussion, Adam generally answered very briefly and evasively. At the presentation of the project, Adam's partner did the talking. At the end of the presentation, the teacher asked them how they felt about this assignment. Adam exclaimed, "Finally we are done with it!", and continued: "I did not like this assignment, I did not quite understand it, and it did not make any sense to me. Then it becomes more difficult to ask for more help."

At the time of this example, the period of online teaching had lasted three months, and Adam clearly felt frustrated and demotivated. About a month later, the students returned to school. The post-data were collected after less than two months of teaching in the classroom, and, as exemplified above, Adam was clearly still affected by his struggles. During the lesson about the sum of angles in a triangle (cf. section 7.4.3), Adam even started crying and left the class accompanied by the teacher, who later told me that Adam was struggling a lot with his well-being. Luckily, he returned shortly after and was able to participate in the mathematical activity. There is, however, one example indicating that Adam might have felt an enthusiasm about returning, that also shows that his frustration in example 1A was not necessarily connected to the type of assignment.

Example 2A: In May 2021, a few weeks after returning to school, the class worked on the project described in section 7.4.4 about planning a party (cf. example 4E). Adam worked with a partner, with whom he was not used to collaborating. I observed the presentations of the projects, and here, Adam presented as very convincing and confident, and the application of mathematics was clearly more advanced than in the previous project. The group even managed to complete extra tasks (doing more than what was required).

Unfortunately, this enthusiasm does not seem to be present in the data collected in June. The question is of course if the shifts in Adam's attitude towards mathematics can be detected in his beliefs about mathematics as a discipline.

Adam's post-beliefs about the application of mathematics

When it comes to Adam's beliefs about the application of mathematics, they seem to have remained unchanged, also when including the data from 2020. In the midway interview, as well as the questionnaire and the post-interview, Adam is unable to exemplify his own experiences with the use of mathematics with anything but shopping. In a societal context, he consistently characterizes mathematics as an aid to keep track of things, although he does not know what these 'things' might be. Nevertheless, in the post-interview, he reflects on what it would be like if people did not learn mathematics:

179	Maria:	What would happen if we didn't learn math in school? […]
180	Adam:	At first, I don't think anything would happen.
181	Maria:	No, but when they grow up, those people who did not learn mathematics?
182	Adam:	Then I think it would start to go worse. And for example economy and things like that would be worse in Denmark.

As seen in the case of Erica, putting mathematics in a context activates Adam's articulation of beliefs. Again, associating mathematics with work initiates more examples:

154	Adam:	[…] if you work at a construction site, then you have to do math, or if you work at a supermarket or something.

In addition, Adam reflects on people's lack of awareness of their use of mathematics:

170	Adam:	You probably use it [mathematics]. You just don't think about it.

In both the midway interview and the post-interview, Adam says that he believes mathematics to be important as it is widely used—so much so that it is impossible to avoid it. Contrastingly, he answers "no" to questions A1 and A2 concerning the importance of mathematics:

A1. *Do you think it is important for you to learn math?* No, *because:* It is only very little of it that you will use after finishing school.

A2. Do you think it is important for everybody in Denmark to learn math? No, *because:* Many of the things you will never use when you grow up, and you forget many of them.

Since the post-interview with Adam was conducted immediately after he responded to the questionnaire, this contrast may be interpreted as an inconsistency in his beliefs. Yet, his answers in the questionnaire might also express his immediate emotions and associations, which were then 'softened' or moderated during our conversation in the interview. It, thereby, becomes an illustration of the importance of context and the influence of the interaction between a student and a researcher when studying beliefs.

In addition to this inconsistency, Adam's beliefs about the application of mathematics still appear somewhat non-evidentially held with hardly any justification. In line with Erica, he does, however, provide examples of the application of mathematics in society, and there are signs of reflection in the post-interview.

Adam's post-beliefs about the historical development of mathematics

In 2019, Adam was able to reflect on issues related to the historical development of mathematics, even though he did not seem to have given it much thought previously. As with the application of mathematics, this has not changed in essence, except for what appears to be a higher degree of certainty or consciousness, since he does not seem as hesitant. Adam still states that mathematics is discovered and thus has existed, at least as long as "human beings learned to think". In fact, Adam gives this answer when asked about the origin of mathematics *learned in school*, and it thereby appears that Adam does not distinguish school mathematics from "real-world" mathematics.

According to Adam, mathematics has changed since its discovery, as it has become more advanced, though he does not know in what way. When asked about whether mathematics is more important now than 100 years ago, Adam answers "yes" in the questionnaire, but does not give a reason for this. Yet, in the post-interview, although still uncertain, he is able to elaborate and justify his answer to this with an example:

201	Maria:	Why do you say yes?
202	Adam:	Because now there are many more things that use mathematics. I don't know… Like, things with energy and stuff.

Although Adam seems to have almost the same beliefs as in 2019, they seem to have somewhat increased in the level of reflection. There are certain indications that the intervention may have contributed to this. In the midway interview, he mentions Pythagoras as an example of what he has learned in school about the history of mathematics, and in the post-interview, he states that the only things he remembers from the last two years of mathematics lessons are "the stories [which can be interpreted as the historical elements] and the pancake activity". In the questionnaire, he mentions Euclid twice:

B4. *What did you learn in school about the history of mathematics?* Euclid

E1. *What do you remember best from the math lessons in 6th and 7th grade?* Euclid

It thereby appears that the activities connected to historical mathematicians have made an impact on Adam, and that they might have played a role in his reflections as elements of knowledge on which he potentially can base his beliefs about mathematics as a discipline. To some extent, Adam's beliefs can be said to have become evidentially held.

Adam's post-beliefs about the nature of mathematics as a subject area

The pattern repeats itself when it comes to Adam's beliefs about the nature of mathematics as a subject area, which are more or less similar to what he expressed in 2019. As depicted in his drawing (figure 29), Adam still associates mathematics with correct results and methods for calculations:

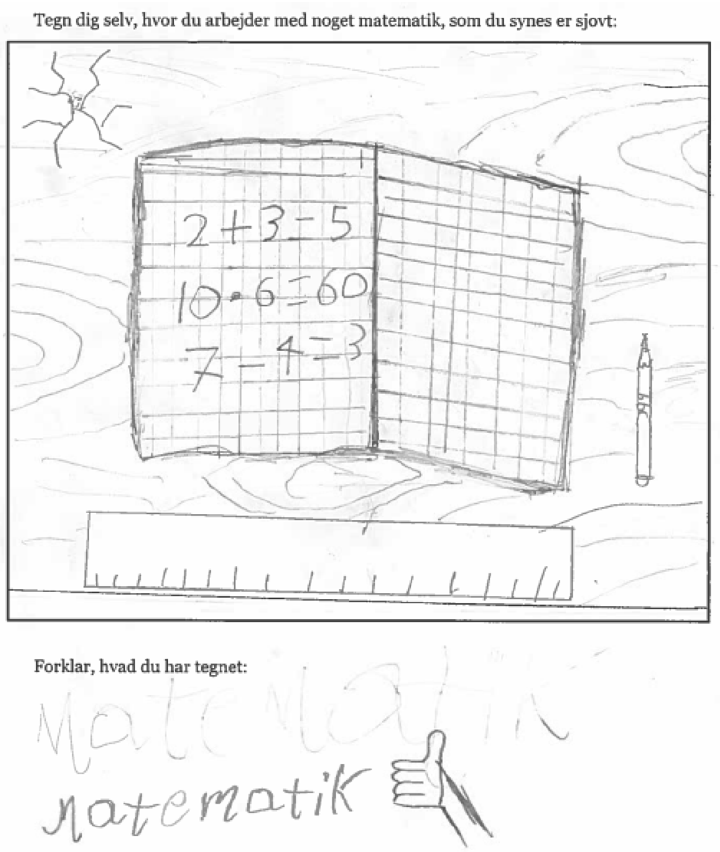


Figure 29: Adam's drawing in question D1, post-questionnaire ("Draw yourself working with mathematics that you find fun"). The explanation below the drawing is "Mathematics".

Compared to other subjects, he finds that the teacher teaches by the smartboard to a higher degree, while the students copy what is written down, and, as mentioned before, he considers mathematics more difficult than other subjects. Like in 2019, Adam begins most of his answers in the interview with "I don't know", indicating a low level of previous reflection on the matter. Interestingly, this is not as pronounced in the midway interview, although the content of Adam's beliefs appears very similar when comparing all three interviews.

Summary

Even though Adam's attitude towards mathematics fluctuated quite a bit during the intervention, it does not seem to have affected the content of his beliefs. His negative attitude in 2021 is primarily reflected in his response to the questionnaire and in some of the classroom observations from the last period of the intervention. However, these data mainly show Adam's immediate reactions and associations, which very likely can be biased by his attitude and motivation towards school in general. In the deeper conversations during the interviews, both in 2020 and 2021, Adam seems to express his *beliefs* to a higher degree. In content, they have not changed, and Adam's ability to reflect, when he is given the opportunity, is still intact. A change *can* be detected, though, in his increased use of examples in 2020 and even more in 2021, especially when related to the history of mathematics, and to some extent also to the application of mathematics. Hence, the drop in Adam's total sum of coding values is most likely related to his decreasing attitude, and not the level of reflection in his beliefs. Furthermore, Adam's positive attitude in 2020 is primarily reflected in his behavior in the classroom, and in the confidence of his answers in the midway interview that appear less hesitant and uncertain.

Chapter 9: Subdiscussion 2

The purpose of research question 2 was to investigate which changes could be detected in the students' beliefs about mathematics as a discipline after a longitudinal intervention with an increased focus on developing the students' mathematical overview and judgment. I have approached this investigation in three steps:

1) Implementing an increased focus on the three forms of OJ through a longitudinal intervention in two middle school classes. The intervention used a Design-based Research approach, in which principles for teaching were designed, implemented, and refined in four iterations, in collaborations with the teachers.
2) Collecting data on the students' beliefs about mathematics as a discipline throughout the intervention. To detect possible changes in the students' beliefs, these data have been analyzed and compared to the results from research question 1. In line with research question 1, this research question has been approached on a general level through a questionnaire, and on an in-depth level through four case studies.
3) Investigating possible correlations between the intervention and changes in the students' beliefs.

This chapter presents and discusses the findings of these investigations as well as the implications connected hereto.

9.1. Findings and implications related to research question 2

9.1.1. Findings: all students

As described in section 8.1, the results of the comparison between the pre-questionnaire and the post-questionnaire point to a rather substantial development in the students' beliefs and their attitudes towards mathematics. In general, the average sum of coding value has increased by approximately ten points, placing the majority of the students in the green category. It, thereby, seems that the intervention has in some way been successful. A large drop in answers assigned code 0 indicates that the students have indeed become more aware or conscious of the issues raised in the questions (i.e., aspects of mathematics as a discipline). Furthermore, their answers include more examples and justifications. Of course, the increased level of reflection may not necessarily be connected to the intervention. The students are almost two years older when responding to the post-questionnaire, and their abilities to reflect and to express their thoughts have very likely improved. Therefore, the *content* of their answers and the character of the *examples* they provide are essential to detect a possible correlation between the intervention and the results. Approximately half of the 38 students mention activities that were designed based on the teaching principles when asked what they remember from the mathematics lessons during the two years of mathematics teaching. Seven students point to such an activity, when answering from what part of the teaching they have learned the most. There are thus indications of a possible correlation, but it cannot be said to be indisputable. Hence, the in-depth investigation may provide results that are more detailed.

Another implication of the general level of the investigation concerns whether or not these results provide an accurate image of the development of the students' beliefs. In the subsequent interviews, the importance of the context in a constructivist perspective (cf. section 3.1) became quite clear, showing that a student's response to a questionnaire is primarily a snapshot that to some extent may be influenced by a number of factors such as their emotions, attitudes, mood, etc. For example, a student may be very tired, and thus not feel motivated to provide elaborative answers, or (s)he might just have had a positive experience with mathematics that is reflected in more positive answers than what perhaps would have been the case the week before. When comparing results from two questionnaires, it, thereby, means that two snapshots are compared, which cannot be expected to precisely reflect the possible development. Nevertheless, the overall comparison may provide an *idea* of possible tendencies in the general development in the students' attitudes and beliefs. As such, the results may serve as a guideline for conducting the interviews with the focus students, both in terms of seeking to confirm or reject these results, and in terms of basing the interviews on the individual students' answers to the questionnaire.

9.1.2. Findings: focus students

In table 39, I have summarized the analysis of the four focus students' post-beliefs, emphasizing where changes can be detected. For the sake of comparison, the findings from 2019 are included (in italics).

Table 39: Summary of analysis, focus students' post-beliefs. Changes from 2019 to 2021 emphasized.

	OJ1	**OJ2**	**OJ3**	**Overall**
Tom Code sum: 51 *(47 in 2019)*	*2019: Math is important and used for everything. Learning math for education and work. Does not see the relevance of learning advanced math.* *Low consistency.* *Few examples (shopping).* *No justification.* *Non-evidentially held.*	*2019: Low consistency.* *Hesitant answers.* *Few examples.* *No justification.* *Non-evidentially held.*	*2019: Math is mostly numbers* *Success criteria: speed, memorization and correctness.* *Low consistency.* *Few examples.* *Low justification.* *Non-evidentially held.*	*2019: Negative emotional associations.* *Relates mathematics to numbers.* *Beliefs about math. as a disc. not yet developed.*
	2021: Content of beliefs unchanged. **Some consistency.** **Higher justification.** Some exemplification. Non-evidentially held. Connection to intervention: Does not feel he has learned about the appl. of math in school, but uses Pythagoras as example.	2021: **Higher level of consciousness.** Low consistency. **Higher exemplification.** **Higher justification.** **Higher level of reflected.** **Somewhat evidentially held.** Connection to intervention: Refers to Pythagoras.	2021: Content of beliefs unchanged. Associates math with numbers and calculations, speed and correctness. Low consistency. **Higher justification.** Low exemplification. **Small signs of increased reflection and changed behavior.** Non-evidentially held. Connection to intervention: changed behavior, primarily in inquiry-based activities.	2021: Beliefs as well as attitude overall unchanged, but signs of higher reflection and potential for development.
Molly Code sum: 61 *(57 in 2019)*	*2019: High consistency.* *Several examples.* *Justification.* *Beliefs based on evidence.* *Evidentially held.*	*2019: Low consistency.* *Few examples.* *Low justification.* *Non-evidentially held.*	*2019: Mathematics is associated with numbers and calculations.* *Sees mathematics as a school subject.* *Success criteria: correctness and speed.* *Evidentially held (experiences from school).*	*2019: Mathematics is primarily seen in a school context and based on experiences in school.*
	2021: High level of exemplification, justification and consistency. Evidentially held. (No clear development) Connection to intervention: None	2021: **Higher level of consciousness.** **Higher consistency.** **Higher justification.** Low exemplification. Non-evidentially held. Connection to intervention: Mentions Pythagoras several times	2021: Content of beliefs unchanged. High level of consistency, exemplification and justification, based on school experience. Evidentially held with experiences from school. (No clear development) Connection to intervention: none.	2021: Slightly more negative attitude. Unchanged beliefs, except about the historical development of math.

Erica Code sum: 57 *(32 in 2019)*	*2019: Math is important for education (exams).* *No feeling of future personal relevance.* *Somewhat consistent.* *Few examples (none related to society).* *Low justification.* *Non-evidentially held.*	*2019: Low consistency.* *No examples.* *Low justification.* *Beliefs not yet developed.* *Non-evidentially held.*	*2019: Mathematics is difficult.* *Success criteria: correctness.* *Signs of change in beliefs from schema-orientation to process-orientation.* *Non-evidentially held.*	*2019: Beliefs are primarily based on experiences in school.* *Beliefs about math. as a disc. are yet to be developed.*
	2021: Content of beliefs unchanged, except about application in society (emerging beliefs) Some consistency. **Higher exemplification**. Low justification. Mostly non-evidentially held. Connection to intervention: Examples from activities in intervention, increased application of math in projects.	2021: Low level of consciousness. Low consistency. Low justification. No exemplification. Non-evidentially held. Connection to intervention: vague indications of having learned about the historical development of math.	2021: Content of beliefs unchanged. Low consistency, exemplification and justification in relation to school math. **Higher level of consciousness, exemplification and justification in relation to math as a discipline.** **Evidentially held to some extent.** Connection to intervention: Variety of examples from intervention in drawing, changed behavior during mathematical activities.	2021: Positive attitude. Not actual change in beliefs but indications of development (more examples), especially in relation to the nature of math as a subject area.
Adam Code sum: 42 *(46.5 in 2019)*	*2019: Math is important and used for everything.* *Few examples (shopping and money), one example for personal experience.* *High consistency.* *Some justification.* *Non-evidentially held.*	*2019: Math has always existed.* *High consistency.* *Few examples.* *Low justification.* *Very few answers (beliefs not yet developed).* *Non-evidentially held.*	*2019: Missing answers (beliefs not yet developed).* *Success criteria: correct results and methods.* *Non-evidentially held.*	*2019: Hesitant or missing answers.* *Beliefs about math. as a disc. are yet to be developed.* *Signs of reflection.*
	2021: **Slightly higher exemplification.** Low justification. Some consistency. Some signs of reflection. Non-evidentially held. Connection to intervention: None	2021: Content of beliefs unchanged. Higher level of consciousness. High consistency. **Higher exemplification.** Low justification. **Evidentially held to some extent.** Potential for change. Connection to intervention: Mentions Pythagoras and Euclid as examples.	2021: Dualistic perspective of math. Hesitant answers. High consistency. Low exemplification. Low justification. Non-evidentially held. Connection to intervention: None.	2021: Volatile attitude (low in 2019, high in 2020, low in 2021). Not reflected in beliefs that are almost unchanged in content, but show changes in higher exemplification (esp. conc. hist.). Still signs of reflection.

Within this analysis, it is worth noting that the *quality* of the students' beliefs—i.e., the level of reflection—may have changed, although they may not have changed significantly in *content*. Table 40 shows a very simplified illustration of how these two types of change are distributed among the students:

Table 40: Change in content of beliefs versus change in quality of beliefs.

	Change in attitude	Change in content			Change in quality		
		OJ1	Oj2	Oj3	Oj1	Oj2	OJ3
Tom	-	-	-	-	+ (Consistency, justification, exemplification)	+ (Consciousness, exemplification, justification, evidentiality)	+ (Justification, reflection, behavior)
Molly	-	-	+	-	-	+ (Consciousness, consistency, justification, evidentiality)	-
Erica	+	-	-	+	+ (Exemplification, evidentiality)	-	+ (Consciousness, exemplification, justification, evidentiality)
Adam	+ (2020) -/+ (2021)	-	-	-	+ (Exemplification)	+ (Consciousness, exemplification, evidentiality)	-

The findings of the in-depth level analysis thereby support the findings at the general level (in relation to an increased level of reflection). As mentioned, this increase may possibly be related to the students maturing during the two years of intervention. There is an expected difference in the ability to reflect of a 12-year-old student and a 14-year-old student. What is interesting in the context of this study is that at least parts of the students' reflections seem to be connected to the content of the intervention. For example, the increased reflection regarding the historical development of mathematics is, in three of the cases, based on experiences with and knowledge about Pythagoras and Euclid. In addition, the data indicate that all four focus students have become more conscious of their beliefs or more aware of the belief objects, as their answers are considerably less hesitant. This supports the interpretation in the analysis of the general tendencies in the two classes regarding the small number of answers assigned code 0.

In two of the cases—Tom and Molly—it is remarkable that the strongest signs of development in their beliefs are found within the aspect of mathematics about which they did not seem to have developed any beliefs in 2019, namely the historical development of mathematics. Also, Erica's beliefs about the application of mathematics in society seem to have emerged. This result supports theory identifying newly acquired and peripheral beliefs as easier to change than beliefs that have become central and robust (e.g. Green, 1971; Pajares, 1992).

It can be argued that the most remarkable development may be found in the case of Erica, as she not only has expanded her perception of mathematics to a large degree, but also seemed to have changed her approach to mathematical activity. To some extent, this also applies to Tom, at least when looking at his behavior prior to the second period of lockdown. Nevertheless, the content of Tom's beliefs has almost not changed at all. It is possible that this difference is related to the learning environment of the two classes. According to Molly, who is in the X class with Tom, as well as the teacher of the X-class, the environment is quite competitive. The students do not feel comfortable making mistakes. From my attendance of lessons in this class, I tend to support this impression. Although Tom does not express this directly, he mentions competitive elements several times. In contrast, Erica is very clear in her description of a safe learning environment, where mistakes are welcomed and the students appear to be supportive of each other. Again, the teacher's descriptions of the Y-class and my observations both confirm this. As mentioned in section 2.4, Goldin et al. (2009) point out that belief change requires an emotionally safe context, which this study thereby supports.

The slightest development can, on one hand, be said to be found in the case of Adam if judging from the sum of coding value in the questionnaire, which actually dropped. On the other hand, the additional data showed a much more complex development, with some progress in his beliefs, but a rather substantial fluctuation in his attitude and emotions causing decrease in the code sum. In comparison, Molly seems to have experienced the slightest development in her beliefs. Apart from the historical development of mathematics, her beliefs do not seem to have changed whatsoever. A possible explication might be related to the centrality of her beliefs. When looking at her beliefs in 2019, her beliefs about the application of mathematics appeared both consistent, exemplified and justified, as well as evidentially held. The same may be said about her beliefs about the nature of mathematics as a subject area, which she clearly equates with mathematics as a school subject, and which she bases on her experiences in the classroom. The high level of reflection, as well as the evidentiality thus make the beliefs central and robust. Furthermore, the above-mentioned change in her peripheral beliefs about the historical development of mathematics supports this explanation.

The question is, however, why the students' beliefs, except in a few cases, have not changed essentially in content. During the intervention, the students have been presented with a number of activities and experiences designed to initiate cognitive conflict. There may, however, be a number of reasons for these experiences not leading to belief changes. One of them may be found in the implementation of the teaching principles in the lessons and activities. To discuss this matter, it might be helpful to examine a lesson that appears to have been successful. The lesson about the Pythagorean theorem seems to have made a lasting impact, especially on Tom and Molly, who are both in the X-class. A closer look at the particular lesson in this class suggests that the inquiry task may have initiated some degree of curiosity in the students and challenged the students at an appropriate level. In addition, the teacher finished the lesson with encaptivating and comprehensive examples of the application and usefulness of the theorem. The lesson thus included concrete examples of the application of mathematics, the historical development, and mathematical methods for inquiry. Furthermore, the students articulated their beliefs about the history of mathematics and participated actively in dialogues about their investigations. However, far from all lessons included the teaching principles to the same extent, and particularly the implementation of reflection seemed to cause difficulties for the teachers. This matter will be elaborated in chapter 10, along with a further discussion of factors hindering possible changes in the students' beliefs.

Even though the focus students do not appear to have changed the content of their beliefs (yet), they all show indications of potential cognitive conflicts. In the case of Tom, we saw that certain types of mathematical activity and challenge caused a change in his behavior that, in fact, conflicts with his beliefs. Molly expressed several examples of contradictory beliefs within the different aspects of mathematics. In the cases of Erica and Adam, providing a context enabled them to activate their beliefs. Unfortunately, the interruptions of the intervention caused by the pandemic-related lockdowns seem to have had a considerable impact on the development of at least some of the students' beliefs. In the first period, the online format was new and challenging for the teachers as well as the students, and the primary goal was simply to provide some kind of teaching, however basic it might be, to live up to the school's teaching obligations, and to preserve a feeling of structure for the students. The intervention was thus set on hold. In the second period, the teachers attempted to follow the teaching principles. However, as mentioned before, several factors seemed to reduce the possibilities for implementing them to a degree that would be comparable to teaching in the classroom (e.g., the teachers' willingness to experiment in an unfamiliar teaching format, or the need to prioritize the well-being of the students). Especially for Tom and Adam, these periods negatively affected their motivation and participation; so much so, in fact, that what appears to be an emerging or ongoing development in their attitudes, learning behavior or beliefs, was set back.

Chapter 10: Discussion

An essential part of Design-based Research is the dual perspective, where theory and practice are included in all phases of a study (cf. section 3.2), and thus also in its contributions. In this chapter, I discuss the findings of this study from these two perspectives and include a third perspective concerning the teachers' role. In addition, I revisit the methodology and evaluate the study based on the quality criteria described in chapter 3. I begin by summarizing the key findings.

10.1. Summary of key findings

As described in chapter 6, there are three key findings related to the students' beliefs at the beginning of the intervention of this study, as examined in research question 1: "*What characterizes middle school students' beliefs about mathematics as a discipline?*":

1. In line with the study of Gattermann et al. (2012), the students in the two participating middle school classes found mathematics important, but they struggled with justifying and exemplifying the reasons for this. This finding thus confirms that the *relevance paradox* (Niss, 1979) still exists. Furthermore, a rather large number of the students seemed to have negative emotional associations with mathematics.
2. As predicted in the literature (e.g., Lester, 2002), the students' beliefs about mathematics as a discipline were to a large extent either non-existent (especially concerning the historical development of mathematics) or similar to their beliefs about school mathematics (especially concerning the nature of mathematics as a subject area). Overall, the reflection level of the students' beliefs was rather low, both in terms of exemplification, justification, and consistency, and most of the beliefs appeared to be non-evidentially held.
3. Consistent with existing research (e.g., Grevholm, 2011; Grootenboer, 2003; Halverscheid & Rolka, 2006; Kloosterman et al., 1996), the students seemed to have a dualistic perspective on mathematics.

In connection to research question 2: *"Which changes can be detected in the students' beliefs about mathematics as a discipline after a longitudinal intervention that focuses on developing the students' mathematical overview and judgment?"*, I wish to highlight five key findings:

4. After two years of implementing an increased focus on mathematics as a discipline, the students in the two classes generally seemed to have an increased awareness of the three aspects included in overview and judgment, their attitude towards mathematics had become less

negative, and they tended to associate mathematics with content rather than emotions. These changes may, to some extent, be linked to the intervention, as a large part of the students referred to activities based on the designed teaching principles.

5. Although the students' beliefs about mathematics as a discipline do not appear to have changed much in content, and the dualistic perspective on mathematics still seemed to be predominant, the students generally seemed to have become more reflected in their beliefs in terms of exemplification, justification, and consistency. To some extent, they also appeared to base them on evidence. Again, some of the examples given by the students can be linked to the intervention.
6. The strongest signs of development were found within the students' beliefs about the historical development of mathematics and the nature of mathematics as a subject area, which were either non-existent or peripheral at the beginning of the intervention. This supports theory suggesting a stronger stability of central beliefs (Green, 1971) when compared to the low degree of development in those of the students' beliefs that appeared to be established and robust at the beginning of the intervention.
7. In line with Goldin et al. (2009), a possible coherence can be found between beliefs development and learning environment, as students expressing a feeling of safety in the classroom seem to have developed their beliefs to a larger degree, as in the case of Erica.
8. Despite the two periods of online teaching, where the teaching principles were not completely implemented as intended, all four focus students show signs of potential development, for example in their behavior or in their reflection level.

Returning briefly to research question 1, the findings connected to research question 2 may in fact provide further information about the character of middle school students' beliefs about mathematics as a discipline:

9. The students' beliefs about mathematics as a school subject are highly reflected in their beliefs about the nature of mathematics as a subject area. Their beliefs about mathematics as a school subject have formed and developed throughout all their school years and have therefore become quite central. This confirms the *perseverance phenomena* (cf. section 2.4). The findings suggest that the more the students' experiences in the classroom support their specific beliefs, the more robust they become. This is partly because they become the basis for other beliefs. Hence, they become accordingly difficult to change, even when presented with contradicting experiences.
10. As the students had only limited beliefs about the historical development of mathematics and did not seem to have had experiences with this aspect, these beliefs tended to be easier to develop, as they did not need to fit into any already established beliefs. Thereby, the students formed their beliefs regarding this aspect rather than developing existing ones.

11. Providing a context for the student' beliefs seemed to activate them and increase the ability to express them.

10.2. Theoretical interpretations of findings

In section 2.5, I argued that the quality of beliefs should be valued on the way they are held in terms of evidentiality and reflectiveness and not only on, for example, their content or their placement within the dualistic/relativistic spectrum. One of the aims of this study was thus to develop the students' beliefs about mathematics as a discipline in a direction based on evidence and reflection. This aim has to some extent been realized. However, this development does not necessarily appear to have made the students *reflective connectionists* (Cooney et al., 1998, cf. section 2.4), as none of the focus students showed signs of attempting to resolve conflicts or differences between beliefs through reflection. At best, some of them (e.g., Adam and Erica) might be said to have become *naïve connectionists*, as they are somewhat able to accommodate and reformulate their beliefs. Still, they were all quite focused on mathematics in a school setting, and did not seem to be aware of any possibly conflicting beliefs. In this regard, it might be argued that the intervention did not manage to evoke cognitive conflicts within the students' beliefs, or that it did not provide adequate opportunities for reflection on potential cognitive conflicts. Possible explanations for this will be elaborated in the following section, when the discussion turns to the challenges connected to the implementation.

Another aim of this study was to increase the students' feeling of relevance towards mathematics through a clear focus on the aspects of mathematics that normally exceed a school setting. In section 1.4, I hypothesized that "a longitudinal change of focus in the teaching of mathematics can contribute to a change in middle school students' beliefs about mathematics – specifically that an increased focus on mathematical overview and judgment can positively influence their beliefs about mathematics as a discipline." With the notion of mathematical overview and judgment, there is a clear orientation towards a relativistic perspective on mathematics. The first aim of (primarily) focusing on the way the students hold their beliefs may in fact be a prerequisite of the second aim of changing the content and the perspective of these beliefs. As stated in chapter 2, beliefs based on evidence are more likely to change through reasoning and reflection (Green, 1971). Thereby, the development found in this study, where the students' beliefs appear more reflected and based on evidence, can very well be the first step towards changing their beliefs in a relativistic direction—or in fact in any desired direction. Somehow, the intervention succeeded in instigating the process of *articulating* existing beliefs, *experiencing* contradictions, and *reflecting* on relations between these, but it failed to complete it. Hopefully, this initial effort can be a starting point that supports both students and teachers in continued development.

Returning to the hypothesis, it cannot be indisputably confirmed. The implementation of perspectives on mathematics as a discipline did not entirely change the students' relations to the subject, and although a positive development in the quality of their beliefs was found, it did not unequivocally influence their perception of the subjects' relevance. With that said, the study does offer several theoretical contributions to research on beliefs.

10.2.1. Theoretical contributions

To begin with, the findings connected to the character of the students' beliefs in this study confirm the results of previous research, as presented in section 2.2.3. Thereby, the study indicates that Danish middle school students do not differ much from students in other countries or, for that matter, from students in other age groups. There seems to be a very clear dualistic tendency in students' beliefs about mathematics as a discipline that exceeds national borders, cultures, and curricula. Searching for the reasons behind, as well as solutions for this must unquestionably be indeed a central endeavor of mathematics education research, as it has already been for years.

Next, it is interesting how the findings show that the intervention has increased the quality (i.e., level of reflection) of the students' beliefs without changing the content. As described in section 2.4, the intervention was partly based on the following process of changing their beliefs (figure 30):

Figure 30: Process of changing beliefs

Included in this process is thus a contradiction of existing beliefs (the middle step) and thereby, experiences that support new beliefs. In that way, it is surprising that this process apparently has made the students *reflect* on their beliefs, but not made them *change* them. It is of course possible that the students in this study did not perceive the "new evidence" as contradictive to their existing beliefs, as they do not articulate the activities during the intervention as contrasting the usual teaching methods. Still, instead of a change in beliefs, the students have overall improved their ability to exemplify and justify their existing beliefs. As discussed above, this may point to an ongoing changing process, where the first steps have been taken towards new beliefs. If that is the case, this study may contribute with a detailed understanding of these first steps of the process. The analyses of the development in the students' beliefs then indicate that an articulation of existing beliefs, along with the first experiences that contradict them, may at first initiate an increased awareness of the existing beliefs. Since a change in beliefs is, as the reader may recall, always the last option (Pajares, 1992), it is possible that the students seek to either ignore the new evidence or attempt to confirm their existing beliefs by increasing the exemplification and justification of these—perhaps even in a way that allows parts of the new evidence to be adapted. A hypothesis may then be that further evidence on the new beliefs, i.e., more experiences that contradict the existing beliefs, will increase the chances of a cognitive conflict. To entail a change in beliefs that is based on evidence, it is then essential that this conflict be solved through reflection.

A third contribution is connected to the *stability* of students' beliefs. As discussed in chapter 9, the findings correlate with Green's (1971) distinction between central and peripheral beliefs and the stability connected to these. Furthermore, the analyses of beliefs of the students in this study confirm the cluster structure of a belief system, enabling contradictive beliefs to coexist. What is particularly interesting is how central the students' beliefs about mathematics already appear to be in 6th grade. The students' encounters with mathematics throughout their lives have indeed formed the lenses through which they perceive new information. In this study, it is clear that most of these encounters have taken place in a school setting, with the norms and success criteria that follows. What makes this even more noticeable is the obvious contrast to the beliefs that have not yet developed. Judging from the collected data, the students had only had few, if any, opportunities to form beliefs about the historical development of mathematics prior to the intervention. When providing them with such opportunities, their beliefs about this matter quickly developed based on the presented evidence. The historical development was thereby the one aspect of mathematics where the students' beliefs seemed to have the highest level of reflection. As a representation of peripheral or newly acquired beliefs, this thus confirms that the stability of such beliefs is much lower than the stability of central beliefs.

What is more, the study also correlates with the definition of, and the distinction between, the three dimensions in the affective domain (McLeod, 1992). Concerning the stability of beliefs compared to the stability of attitudes, the findings show that where the students' beliefs generally did not change in essence, some of the students' attitudes towards mathematics (as a school subject) clearly shifted during the intervention, as seen in the cases of Erica and Adam. In addition, Erica's mood in the post-interview clearly illustrated the instability of emotions. The attitude framework by Di Martino and Zan (2010) quite effectively identifies how the three aspects of the affective domain interrelate. By including 'vision of mathematics', which relates to beliefs about mathematics, and emotional disposition, which of course is connected to emotions, the framework emphasizes the placement of attitudes in the 'middle' of these three, thus influenced by the other two. Judging from the present study, the influence of attitudes and emotions on a person's beliefs primarily seem to be indirect. The students' beliefs remain stable even when attitudes and emotions change, but their openness to new experiences and to meeting new information may be affected. This might have been the case with Tom and Adam during the second period of lockdown. Thereby, the findings indicate that attitudes and emotions can potentially influence the changing process of beliefs, but not necessarily on already existing beliefs.

Finally, a fifth contribution is related to the *complexity* of students' beliefs. The analyses show numerous examples of how the students' beliefs are dependent on and intertwined with other beliefs and their relations, as also pointed out by Rolka and Roesken-Winter (2015). Clearly, the other dimensions of the belief system strongly affect the students' beliefs about mathematics as a discipline, as for example seen in the prevalent influence of their beliefs about mathematics education or the impact of their beliefs about the socio-mathematical norms in the classroom. Tom's experience with mathematical inquiry during the investigative activity connected to the Pythagorean theorem could potentially have initiated a cognitive conflict, as this form of work was contradictive to his beliefs about mathematics as procedural, oriented towards fast results and, to a large degree, pointless. This activity provided Tom with experiences connected to all three forms of OJ, and it certainly seemed to have made an impact on him. Nevertheless, he subsequently managed to accommodate this to fit his existing beliefs about himself as indifferent, but competitive and eager to finish, as he excused his perseverance with boredom, and his eagerness with competitiveness. Tom's beliefs about himself seemed to be so central and robust that they might have hindered a cognitive conflict and instead made him adapt the new experience to his existing beliefs. An accommodation of his existing beliefs towards perceiving mathematics as relevant and useful would have conflicted with his indifference, forcing him to make an effort. Similarly, he would not have been able to uphold his emphasis on fast results when applying characteristic mathematical processes such as inquiry or modeling. Tom perceived himself as good at performing a procedure and finishing quickly, and a more relativistic perspective, therefore, might have caused a change in his beliefs about himself as being skilled. Likewise, Molly's beliefs about mathematics as a school subject seemed to be so central that they also represented her beliefs about the nature of mathematics as a subject area (OJ3), and to some degree shaped her emerging beliefs about the historical development (OJ2). These findings point to the importance of considering all dimensions of the students' belief system when seeking to change their beliefs, even if the focus is on only one of them. Of course, this makes it even more complicated to change students' beliefs and to understand the processes and factors involved herein.

10.2.2. And it makes me wonder ...

After completing this study, the above accounted for theoretical contributions and considerations thus generate a new set of theoretical questions:

- Students in different parts of the world, of different cultures and age groups, seem to possess similar beliefs about mathematics as a discipline, generally perceiving it from a dualistic perspective. What might be the reason for this, and how can we change it?
- What roles do attitudes and emotions play in the process of developing or changing beliefs? And how do beliefs influence attitudes and emotions?
- How do the four dimensions of students' mathematics-related belief systems (beliefs about mathematics education, social context, self, and mathematics as a discipline) and their interrelations influence the process of belief change?
- Is it possible to identify the process of changing beliefs in more detailed steps? If so, which steps should such change, at least, contain? Also, what influential factors can be identified in this process?

10.3. Teacher perspectives

Before I revisit the methodology, I turn to the teacher's participation in this study for a moment. Although the focus is on the students and their beliefs, the teachers have played an essential part in the intervention. Along the way, it became clear to me that certain issues have been so influential for the results that they, necessarily, must be addressed in a section of their own.

It is important to me to underline the priceless contribution that participating teachers have made to this study. They have contributed to the project with great enthusiasm, good ideas and exciting input for the practical execution. The two teachers have very different prerequisites and backgrounds, and they have been involved in the project in different ways. The teacher in the Y-class has been able to draw on her long experience and education, as well as continuously contribute with suggestions for the execution of the teaching. The teacher in the X-class has welcomed all suggestions and ideas and asked many questions that have led to important didactical discussions and considerations. Hence, they have both generously contributed to a successful intervention with their respective competences.

However, unavoidably and despite all the qualities of the two teachers, the gap between the intended and the enacted intervention was at times rather large, which may, of course, also have affected the results, at least in terms of trustworthiness and generality. For example, the implementation of the element of reflection was for some reason problematic, and the classroom discussions did not always proceed as intended. There may be several explanations for this. In the following, I discuss these implementation-related challenges from two perspectives. Firstly, I turn to implementation research (IR), which might provide helpful theoretical constructs or frameworks (cf. Østergaard & Jankvist, In press). Secondly, I draw on a large part of the literature already described in this study, namely the research literature on teachers' beliefs.

10.3.1. Searching for answers within implementation research

In their often-cited journal article, Century and Cassata (2016) list six factors that might influence the implementation of an innovation: spheres of influence, characteristics of end-users, organizational and environmental factors, attributes of innovation, implementation of support strategies, and implementation over time.

In this attempt to explore possible explanations for the recurring discrepancy between the intention and the enactment of the intervention in this study, I address three of the six factors, which appear relevant in this context: characteristics of end-users (here: the teachers), attributes of innovation (intervention), and implementation of support strategies.

The **characteristics of the teachers** can potentially be essential in an intervention process, especially when the intervention gives room for the teachers' interpretations and adaptations. Individual competence, prior knowledge, feeling of agency, and professional identity are all examples of influential factors related to the teachers. Century and Cassata (2016) distinguish between characteristics *existing independently of the innovation*, "e.g., willingness to try new things, organizational skills, classroom management style, or views about teaching and learning in general" (p. 185); and characteristics *related to the innovation*, "e.g., level of understanding, expertise, prior experience, beliefs, values, attitudes, motivation, or self-efficacy" (p. 185). In this regard, Rogers (2003) identifies two different forms of knowledge concerning the implementation: *how-to knowledge*, (practical knowledge about how to apply the innovation), and *principles-knowledge* (understanding the thoughts behind the innovation). In the present study, the characteristics of the two teachers were quite diverse, as described in section 3.2.2. Where the teacher in class X could be described as an almost novice who was trying to find her way in the job, the teacher in class Y was an expert teacher, with a strong professional identity, who often advised her less experienced colleague on mathematics teaching and learning. Nevertheless, they were both highly motivated and found the intervention relevant for their teaching, for the students' learning outcome, and as part of their professional development. In all phases of the intervention, the two teachers clearly indicated a high level of principles-knowledge, in particular the Y-class teacher. During planning sessions and debriefing meetings, we had fruitful didactical discussions, where we agreed on the purposes and intentions behind the intervention and sought solutions for any doubts or struggles. However, as some of the examples in the case analyses in chapter 8 show, the teachers might not have had a sufficient how-to knowledge, and the intervention did not include strategies for developing this, as will be elaborated below when addressing the support strategies. In addition, the large difference between the two teachers may have contributed to the discourse of our sessions, as much time was spent accommodating the quite different needs of the two teachers. Where the teacher in the X-class primarily requested a form of counseling to support her in establishing her teacher identity, dealing with students and parents, or designing activities (how-to knowledge), the teacher in the Y-class was in some way more interested in general didactical considerations. Still, the Y-class teacher also took on her role as a mathematics counselor or a form of mentor for the X-class teacher. In some way, the discussions included theoretical issues as well as practical issues, but the link between the two was rarely established.

The **attributes of the intervention** concern both the subjective teacher perceptions (e.g., relevance or ease of use) objective, and characteristics of the intervention (e.g., number of components, design features, or level of explicitness). Where some interventions are rather explicit, specifying the content and procedures in detail, others are more ambiguous. In this intervention, the level of explicitness was quite low, as it was based on a few rather general principles. Furthermore, it was very ambitious in its intention of a teaching approach, which differed a great deal from what both teachers and students were used to. The collaborative planning and the theoretical, didactical, and practical discussions and decisions were only parts of the implementation. When it came to the actual realization of the principles and the enactment of the intentions, it was all in the hands of the teachers in the classrooms. Thereby, the implementation was highly reliant on the teachers' interpretation and adaption of the principles, requiring a high level of how-to knowledge as well as principles-knowledge. Another attribute of the intervention was the allocation of contributions. As mentioned in section 3.3.4, my primary function as a researcher was to be an expert on the theory, and the teachers primarily functioned as experts on practice. The teachers thus had the primary responsibility for the detailed planning and preparation of lessons. On one hand, this enabled them to adapt the teaching to their individual approaches as well as to the two classes (X and Y). On the other hand, it meant that my control of the actual implementation was reduced, thereby increasing the risk of non-intended realization. Hence, the support strategies became essential.

According to Century and Cassata (2016), **support strategies** should ideally be included in the design of an intervention, based on underlying theories. Supporting the teachers in a process of implementation may appear in various formats (e.g., professional development, strategic planning, or evaluative processes). To assist the teachers in this study, several theoretically based formats of support were included. For example, the theoretical concepts behind the intervention were thoroughly discussed to enhance the teachers' principles-knowledge. We planned some of the lessons in detail together and we held debriefing sessions after the lessons if practically possible. Furthermore, the two teachers often collaborated on their daily planning. These support strategies were further developed during the study, so that I increased my participation in the detailed planning, as I became more and more aware of the challenges connected to, especially, the teachers' how-to knowledge. The teachers, who found it beneficial to engage in the concrete as well as more abstract discussions about the implementation, welcomed this initiative. Nevertheless, most of our discussions, especially in the beginning of the intervention, revolved around the principles-knowledge. This perhaps led us all to think that our intentions and ideas of realization were aligned. The classroom observation indicated that this was not always the case. In particular, the element of reflection caused the teachers problems. An example is the lesson described in the case of Molly (example 1M), where the teacher omitted all planned elements of reflection. What is interesting is that the teacher herself suggested and argued convincingly for all these elements during the planning of the lesson, but still did not include them in the enacted intervention. The general difficulties connected to implementing reflection in the teaching was addressed in every retrospective analysis, where the teachers expressed their struggles. We thus attempted to concretize the teaching principle concerning reflection. What was not included in the support strategies was an actual strategy for professional development. As the study focused on the students' beliefs, addressing the teachers' competences or prerequisites were not prioritized in the design, and the initial information about key elements seemed adequate to ensure that they understood the principles behind the intervention. Still, one might argue that the theoretical constructs behind this study are all rather ambiguous and perhaps difficult to concretize in practice. Concepts such as *beliefs*, *overview and judgment*, and *reflection* can all be perceived and discussed on what might be termed as a general, theoretical or an abstract level, but it is not easy to transfer them to concrete learning activities. Furthermore, there is a high risk of assuming a common understanding of these concepts. A deep understanding of the concepts—regarding both their theoretical aspects and their importance to students' learning and the intention of the study—is crucial to the success of the implementation. As such, an increased effort related to the professional development of the teachers in advance of the intervention, would very likely have been beneficial for the study. Such effort might for example be in the form of workshops, with intense studies of the theoretical constructs and their importance to students' mathematical learning, in-depth discussions of how the theoretical framework could be applied in practice, or detailed cases of exemplary learning activities.

The combination of the above three influential factors points to interesting discrepancies in the design of the study in terms of the implementation process. Firstly, the characteristics of the teachers influenced the dialogue between the three of us, as their very different backgrounds and requirements somewhat divided our discussions into two levels that were rarely combined. Secondly, the low explicitness of the intervention entailed high demands on the teachers' interpretation, adaptation, and enactment of the intervention. This increased the importance of ensuring both their principles-knowledge and how-to knowledge. However, an adequate effort to ensure this was not included in the design of the study. Thirdly, the allocation of expertise complicated the support strategies that could, to some extent, have accommodated this problem, as there was a discrepancy between ensuring the teachers' feeling of agency and ensuring the intention behind the intervention. As a researcher, I therefore often found myself in a dilemma, when I observed deviations from the intention in the classroom and had to address these in the debriefing. It was, of course, essential to the success of the intervention that the intention be realized. Conversely, I had to preserve a respectful communication in terms of the teachers' respective areas of expertise, while at the same time supporting and benefiting future cooperation and innovation. Addressing problematic issues related to practice thus sometimes became a difficult act of balance that I often accommodated by inviting the teachers to share their experiences of the lessons. Our discussions were based on these.

The application of constructs from implementation research reveals some extent of incompatibility between the attributes and overall aims of this intervention. The aim of developing the students' beliefs through designed teaching principles may be too ambitious in terms of *the culture of practice,* which is deeply rooted in the teachers' beliefs about mathematics and mathematics education.

10.3.2. Teachers' beliefs

There is no doubt that the beliefs of teachers are an important influential factor, not only in the teaching of mathematics, but also in an implementation process and in the development of the students' beliefs. As the reader may recall from chapter 2, the formation processes of beliefs are of a general nature, also applied to the beliefs of teachers, who as mentioned often carry their beliefs about mathematics with them from as far back as primary school (Uusimaki & Nason, 2004). A teacher's beliefs is one of the key elements that Ernest (1989) identifies as essential to the practice of teaching along with the social context and the level of thought processes and reflection. In the beliefs of teachers, Ernest includes their conception of three aspects: the nature of mathematics, the nature of mathematics teaching, and the process of learning mathematics. Several researchers have pointed to a necessary distinction between a teacher's *espoused* and *enacted* beliefs (e.g., Cooney, 1985; Eichler, 2011; Furinghetti, 1996), as these often do not match. For example, some teachers may be influenced by the curriculum in their peripheral beliefs, while having deeply rooted beliefs about teaching that are not compatible, resulting in what Furinghetti (1996) terms "ghosts in classrooms". In the present study, there are several examples of this. One of them is the nature of our didactical discussions in the planning sessions which, as explained above, seldom succeeded in concretizing the theoretical concepts and the didactical goals. The teachers usually expressed a problem-solving view of mathematics (cf. section 2.2.1) in the didactical discussions and in the overall planning of the lessons. However, according to Ernest, a problem-solving view can be linked to perceiving the teacher as a facilitator and learning as active construction of understanding. Still, despite comprehensive discussions related to this in the planning sessions, the teachers often took the role of instructor or explainer in the enactment of the intervention. These roles are to a larger degree related to an instrumentalist and Platonist view of mathematics (Ernest, 1989), respectively.

Ernest (1989) identifies two key causes for the discrepancy between espoused and enacted beliefs: the influence of the social context and the teacher's level of consciousness of his/her own beliefs. The social context influences the teachers' beliefs through, for example, expectations of others (students, parents, colleagues, etc.), curricular restraints, systems of assessment, and educational traditions and culture. In the present case, the teachers—as most teachers do—often considered how the content of the intervention might match the requirements for passing the final examination after 9th grade. They were also very attentive to the expectations of the students, as they often took into consideration what the students were "used to" or would think of as "real mathematics". In particular, the X-class teacher was undoubtedly influenced by expectations from parents, colleagues and, not least, the school leader, as she was new to the job. In contrast, the Y-class teacher might have felt certain expectations for her to be in some way progressive in her teaching, being a mathematics counselor. Because of her long experience, she, furthermore, seemed to have strong and central beliefs, not only about mathematics and mathematics education, but also about herself as a mathematics teacher. As mentioned above, the intervention's support strategy did not include an explicit facilitation of the teachers' consciousness of their own beliefs. Perhaps an increased focus on this consciousness, as well as on the teachers' level of reflection would have contributed to an increased alignment of the intended and the enacted intervention.

The study thus confirms the importance of the teachers' beliefs in an implementation process. In some way, the designed teaching principles of the intervention can be perceived as a form of added curriculum. Thereby, the model in figure 31 may be used to illustrate the phases involved in the path from the intentions behind the principles towards the students' learning. The teachers' interpretation of the principles (curriculum) is influenced by those of their beliefs that they take into account during the planning of their teaching, resulting in what can be termed an intended curriculum.

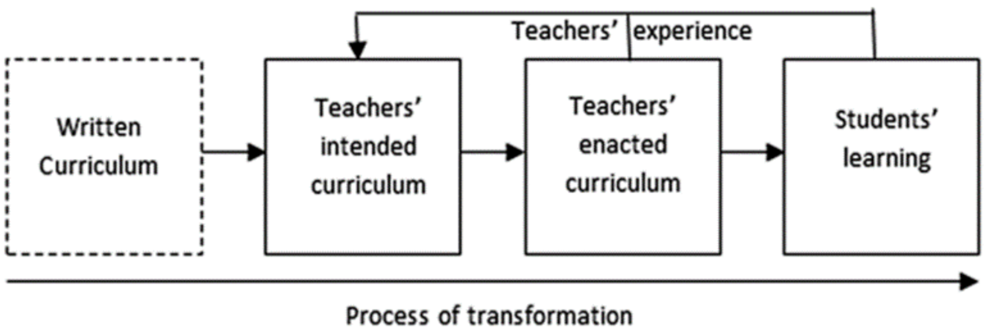


Figure 31: Four phases of the curriculum (Eichler, 2011, p. 176, adapted from Stein et al., 2007).

When realizing these intentions in the classroom, their enactment is, as mentioned above, influenced by the social context as well as by any potential "ghosts in the classroom". Hence, there is a long way from intention to outcome, and a considerable part of this way is dependent on the teachers and their beliefs.

According to the model in figure 31, the teachers' experiences in the classroom might contribute to their interpretation of the curriculum, or, in this case, the teaching principles. This study offered the teachers opportunities to affect the principles through ongoing discussions. Interestingly, they did not make much use of these opportunities, as described in chapter 7. However, a search for the reasons behind this is too extensive to be included here.

10.3.3. Possible issues to be investigated

Although this study focuses on developing students' beliefs, the issues raised in this section distinctly point to the crucial role of the teacher in this process. The question is, if a targeted development of students' beliefs is at all possible without including a certain attention to the teachers' beliefs and their possibilities for enacting and implementing the intentions behind a research design. In this regard, some questions emerge:

- How can implementation research inform and contribute to increase the teachers' readiness, competencies, and opportunities so that the gap between the intended and the realized intervention is reduced (to an extent that still leaves room for updating and adapting of the design when necessary) (cf. section 3.2.1), in attempting to develop students' beliefs (about mathematics)?
- What relationships can be found between teachers' beliefs about mathematics education and the *principles*- and *how-to knowledge?*
- Do the teachers' beliefs about mathematics as a discipline and mathematics as a school subject correspond to those found among the students?
- In what ways would a similar study be able to incorporate initiatives to support the teachers' reflection level and consciousness of their own beliefs?

10.4. Revisiting the methodology

10.4.1. Methodological contributions, implications, and limitations

Approaching an inaccessible construct

Inherent in the epistemological constructivism lies the premise that there is no such thing as an "objective reflection" of a person's beliefs (cf. section 3.1), as expressions of beliefs are always dependent on the context and influenced by whatever activates them. In this study, several of the focus students expressed their beliefs in quite different ways, for example, when comparing their answers to the questionnaires and the interviews, or when provided with a form of activating context. Sometimes, these expressions even pointed to contradictory beliefs. As described in chapter 2, the cluster structure of belief systems enables contradictory beliefs belonging to separate clusters, which may be activated in different contexts (Green, 1971). The question is, if only the *expressions* of beliefs change with the context, or if the beliefs *themselves* are dependent on the context. If the latter is the case, it might be argued that beliefs are not as stable as presumed, and the measuring of beliefs will in fact be influenced by the context to the degree where it is almost impossible to infer anything generalizable about students' beliefs. These concerns are taken into account in the design of this research through the longitudinality of this study, along with the triangulation of data types. Studying the development of the students' beliefs for a certain period of time made it possible to compare the students' actions and utterances at different stages of this period, as well as across different types of data collected simultaneously. In this regard, the findings show that the students' beliefs tended to remain more or less stable over time, as described in chapter 9 and earlier in this chapter. Furthermore, the longitudinal design enabled relating changes in the students' emotions and attitudes to their expressions of beliefs, and, as discussed above, the findings suggest that these changes might be what caused differences in the *expressions* of their beliefs—not necessarily in their *beliefs*. Where the longitudinality enabled a chronological comparison between phases, the triangulation of data types enabled a comparison of different contexts and different sources of activation. Studying the students' expressions of their beliefs in the questionnaires, the interviews, and the classroom observations played an important role in approaching their beliefs. Even though these expressions were sometimes different and inconsistent, the triangulation made it possible to give a nuanced image of the relationships between the students' beliefs, for example, concerning the centrality of these. It also provided opportunities to analyze and infer the influence of specific contexts and identify what was and what was not consistent in the beliefs, when the context as well as the students' attitudes and emotions changed. This furthermore contributed to investigating the level of reflection.

Impacting the development of students' beliefs

One of the methodological aims of the study was to investigate how beliefs can be developed and by what means. In the cases where signs of development can be detected, it seems that the process for beliefs development applied in the intervention (in short: awareness—experience—reflection) may in fact contribute to developing the students' beliefs. Principles for teaching that focus on the students' overview and judgment may function as a guideline for teachers' planning, but it is the didactical details of the activities that seem to cause an effect on the students' beliefs. The three forms of OJ can contribute to the content of the teaching, but if the students' beliefs are to be developed or formed, elements of *awareness*, *cognitive conflicts* and, not least, *reflection* are essential. In light of the above-mentioned implementation-related challenges, this study indicates that promoting these elements in the teaching might require a change in the general teaching approach and the perspective of mathematics. However, these changes need not be immense. As for example seen in the case of Tom, an appropriate challenge, or a specific question instigating curiosity, may sometimes be adequate.

DBR as an approach to study beliefs development

Certain implications were connected to using a Design-based Research approach in the study. One of them has to do with one of the main learning goals in the hypothetical learning trajectory (Bakker & van Eerde, 2015): to develop the students' beliefs about mathematics as a discipline through an increased focus of their mathematical overview and judgment. The nature of this goal is somewhat abstract, and it may be very difficult to do thought experiments and set up hypotheses for the outcome of the intervention. This is for at least two reasons. First, mathematical overview and judgment is a complex construct, and it involves many aspects and perspectives, which all can be discussed and interpreted according to one's perspective on mathematics. Second, it involves knowledge *and* beliefs. Since beliefs, as discussed above, are both difficult to access and to assess, a formulation of a hypothetical learning process (cf. chapter 7) would entail an articulation of steps within a progress in the students' beliefs. However, as beliefs are indeed individual and may be activated to different experiences and contexts, such a formulation would on one side be as abstract as the learning goal and on the other side have to be general enough to fit all students (and thus not concrete enough to be useful). This condition became clear during one of the first planning sessions with the teachers, where we decided to omit this part of the hypothetical learning trajectory. Nevertheless, we did not experience a need for it during the intervention, as the learning goals and the learning activities sufficed within the design, the implementation, and the evaluation/redesign phases.

The complexity of the overall focus of the intervention was also consequential to the implemented *design.* Traditionally, the design of an intervention constitutes one of the pragmatic contributions of a DBR study. It plays a central role in the study, as it consists of certain initiatives or activities that are designed with a specific purpose in mind, and then tested and refined in iterations. In this study, it is the *purpose* (to develop the students' beliefs) that plays the central part, and the designed teaching principles are in essence a means to fulfill this purpose. As the purpose is so abstract, the principles become equally abstract and dependent on interpretations. Hence, the tested and refined teaching principles must not be perceived as a "how-to guide" in relation to developing students' beliefs or implementing the three forms of overview and judgment. Still, the principles may contribute with didactical considerations and ideas about which elements are necessary or recommendable in a competency-oriented teaching approach, where mathematical overview and judgment is taken into account. In addition, the learning activities may also inspire to ncrease focus on OJ.

Another implication concerns the retrospective analyses between the iterations, which too were affected by the abstract learning goals, and thereby the object of these analyses. We have already established that beliefs develop slowly and resistantly, that this process is complicated to monitor, and that they are difficult to measure. Therefore, it was not possible to perform the retrospective analyses in terms of the students' beliefs. It would simply be premature to analyze any changes in their beliefs during the intervention. This was confirmed with the midway interviews, which, at the time they were conducted, did not point to any remarkable changes, but when applied in the analyses of the students' post-beliefs, contributed to a more detailed picture of the development not only in the students' beliefs, but also in their attitudes, as in the case of Adam. Hence, the retrospective analyses were instead mainly based on the teachers' experiences with and evaluation of applying the teaching principles. Furthermore, the impression that the teachers and I had of the students' responses to the teaching and of potential signs of belief development was also included in the retrospective analysis. The adjustments and refinements of the principles were thereby highly directed towards the teachers' feeling of applicability, and based on comprehensive discussions regarding theoretical and pragmatic concerns, as well as interpretations of the students' behavior.

With that said, the DBR approach has made it possible to study the students' beliefs in the context within which they are formed and developed, namely through experiences in the classroom. Furthermore, it has enabled an extensive and valuable collaboration with the teachers. During as well as after the intervention, both teachers expressed that the collaborative planning and didactical discussions with the presence of a researcher in mathematics education had been very beneficial for them, and almost served as professional development. They both stated that it had raised their awareness of aspects in mathematics and of mathematics education to a level that would not have been reached without this intervention. For me as a researcher, the collaboration has been of enormous importance in relation to all practical perspectives on the intervention, as well as to any issues regarding the students.

Implications connected to the students' age

Since the participants in this study were teenagers and thus in a very sensitive and turbulent phase of their lives, their age may have influenced the findings of the study in relation to several issues. One of them is that their personal development may entail a natural increase in their level of reflection. This has been discussed in chapter 9. Another issue is their motivation. Teenagers generally start questioning authority, including their teachers and school in general. Furthermore, they are often preoccupied with their identity, social positioning etc.; mathematics in particular and schoolwork in general may not seem relevant to them at this age. All four focus students in this study expressed a decrease in motivation, and it is not possible to establish an unambiguous reason for that. There is a good chance that it is related to their age, but it may also be related to the periods of lockdown or even to the intervention, although the data shows no indications of the latter.

10.4.3. Potential further considerations

A study of the development of students' beliefs may be conducted in a number of methodological ways. The choices made in this study have led to certain findings, while other choices might have led to different contributions. Hence, the discussion of the methodological choices initiates further considerations:

- The longitudinality of this study and the triangulation of data types have enabled a study of the stability of students' beliefs in different contexts and through different sources of activation. It might be interesting to investigate whether a focused and designed use of context change and activation sources can contribute to a development in students' beliefs.
- The broadness and generality of the learning goal concerning overview and judgment was consequential to the hypothetical learning trajectory, the retrospective analysis, and the applicability of the pragmatic outcome of the study (the teaching principles). It may be worth considering the possibility of setting more concrete and adaptable goals in relation to mathematics as a discipline.
- A sample in a younger age group may not be able to express their beliefs and thoughts about mathematics as well as the students in this study. Still, this study clearly shows that beliefs are formed, confirmed, and strengthened from the early school years. In addition, these early formed beliefs act as filters for new information. Hence, there are good reasons to address and investigate the development of young students' beliefs about mathematics as a discipline.

10.5. Evaluation of the study – returning to the quality criteria

10.5.1. Trustworthiness

Trustworthiness of qualitative research is, as described in section 3.5.1, connected to a high level of descriptive and explanatory power, which can be reached through rich, thick descriptions, focusing on precision and specificity in all steps of the research (Schoenfeld, 2007a).

The descriptive power of this study may be expressed in terms of the applied theoretical constructs concerning beliefs and their capacity to focus on the issues in focus, namely the students' beliefs about mathematics as a discipline and the development hereof. As discussed in section 10.2, the definitions related to the affective domain in general, and to belief systems in particular, have indeed provided a highly applicable and appropriate framework for the research design as well as the data collection and the analyses. The identification of essential dimensions of beliefs has enabled a focus on the mechanisms in the development of students' beliefs. For example, the notion of central versus peripheral beliefs was essential to issues concerning the stability, and thereby the changeability, of beliefs. The explanatory power lies in the possibility to determine potential correlations between different aspects of beliefs, between beliefs and other constructs, and between data and conclusions. As discussed above, the triangulation of data has especially contributed to this, and the longitudinal and comprehensive data collection has made it possible to study the students' beliefs from several perspectives, in different contexts and within time intervals. *Context,* especially, has played a highly important role in this study in terms of the investigation of beliefs, and the implementation of the intervention.

As already discussed, there are many uncertainties connected to the findings of this study, primarily because of the interpretive nature of beliefs and the complex context of the classroom. Thereby, the soundness of the conclusions is, of course, equally debatable. This is, however, typical for qualitative research, not least within educational research concerning affective aspects. The soundness of the conclusions thus lies in the clear and detailed descriptions of how these conclusions are drawn, what data and presumptions they are based on, and which implications there may be. Throughout the study and this dissertation, I have sought a high level of documentation and precision, as well as transparency in the theoretical and methodological choices made.

Another aspect connected to the trustworthiness of the study is my own potential bias as a researcher. In retrospect, one of the possibilities mentioned in section 3.3.1 may in fact have influenced the results. It concerns my empathy with the teachers and relates to the dilemma described in section 10.3.1 about the allocation of expertise versus ensuring the intention of the intervention, and is in some way also part of the ethical considerations. A successful implementation of the intended principles is of course crucial to the study, but it is on the other hand highly dependent on the teachers' feeling of agency as well as professional benefit. In this way, these two aspects are inextricably related. Yet, in some situations, as exemplified earlier, they might be in conflict. Looking back in the project log, my professional background as a teacher may in fact have influenced some decisions, in terms of a strong understanding of the teachers' situation and an attention to a successful collaboration. However, as pointed out in section 3.3.1, I have paid specific attention to this possible bias, and have regularly consulted more experienced researchers regarding this matter.

10.5.2 Generality

In addition to concerns related to the trustworthiness of the study, the quality of qualitative research also lies in its *generality*, which may be considered in relation to a number of aspects.

One of these aspects concerns the extent to which the findings are restricted to the specific context. If the study is to contribute to the field of mathematics education in any theoretical, methodological, or pragmatic way, the findings should somehow include generalizable features. The question here is whether a similar intervention in a different context, for example in other Danish middle school classes, would lead to similar findings. Some unique conditions speak against the generality of the study. One is, of course, the circumstances related to the pandemic situation, which caused long periods of online teaching, challenging the teachers' work, the implementation of the principles, the students' motivation, and the data collection. Still, unforeseen events can occur in every context as complex as a classroom, not least in a longitudinal study. Although the periods of lockdown might have influenced or diminished the findings, it does not exclude the possibility of generalizable features in the findings.

As described in section 3.3.2, the students and the teachers may be said to be unique. This is supported by the findings related to the two classes, which show that the developmental pathways are not similar. The general impression from the data of the two questionnaires shows a progress in the level of reflection and/or attitude towards mathematics, as well as an equalization in the average coding values of the two classes in all aspects of mathematics as a discipline. However, the X-class shows particular development within the aspects of the nature of mathematics as a subject area, where the Y-class seem to have developed more in regard to the historical development. Hence, there are strong indications either that the intervention has not been implemented similarly in the two classes, or that it does not lead to the same results in different contexts— or both. The classroom observations confirm the former, which may be related to the difference between the teachers, the students, or the learning environments. It is thus highly unlikely that a similar intervention would lead to the exact same findings. Nevertheless, the participants represent typical classes, students, and teachers in an average Danish middle school. The dissimilarity between the teachers, and the diversity in the selected focus students in terms of attitudes and beliefs (cf. section 4.3) provide a nuanced representation of potential outcomes, challenges, reactions, and implications connected to the intervention. Thereby, the thick descriptions of the possible correlations between the intervention and the data, as well as between the data and the conclusions concerning the character and the development of the students' beliefs, contribute to an understanding of the perspectives and mechanisms involved. They, furthermore, make it possible to compare this context, these participants, and these circumstances to others, and thus assess the generalizability of the findings. Although the findings are indeed connected to the context, the two classes exemplify a diversity of both teachers and students, which is recognizable in a Danish compulsory school context.

Furthermore, the findings connected to the character of students' beliefs appear to be equivalent to existing research, thus indicating that there is a *general* tendency in students' beliefs. Thereby, other students in middle school and other age groups are likely to possess some of the same beliefs as the students in this study. This is especially true in terms of Danish students, as they share traits of the same educational tradition, culture, and curriculum. The findings thus contribute to a general understanding of possible challenges within a competence-oriented teaching and in connection to developing students' beliefs in general and about mathematics as a discipline in particular. In addition, the potentials lying in an increased focus on students' beliefs and on their mathematical overview and judgment may also be transferred to other contexts.

Another aspect of the study's generality concerns the extent to which the intervention would be *transferable* to other contexts, i.e., if it would be possible to implement a similar design, for example, in other Danish middle school classes. If so, the range of the transferability might be considered: would the intervention design work among other age groups, or in other countries? In light of the above-mentioned general tendency in the character of students' beliefs about mathematics as a discipline, one may argue that these prerequisites, to some extent, are general among students in Danish middle school, and that the 'starting point' for similar interventions would be somewhat comparable. In addition, the cultural context as well as the shared curricula, increases the generality of the design and relevance of the intervention in similar contexts. With the rather general nature of the learning goals and the teaching principles, they are not attached to any particular age group, stage of learning or certain prerequisites, but can in fact be perceived as adaptable to a variety of these factors. And as the notion of mathematical competence—including mathematical overview and judgment—is considered part of an individual's *Allgemeinbildung* in all levels of Danish mathematics education, there are reasons to argue that the intervention may be applicable in whatever context it is considered relevant. As the character of students' beliefs about mathematics as a discipline seem to exceed national boarders, it can further be argued that the intervention may also be relevant for students in other countries with similar aims to increase students' mathematical competence as part of their *Allgemeinbildung.*

Furthermore, the theoretical properties of students' mathematics-related beliefs and the development of these are not restricted to apply only to middle school students. Overall, existing research on beliefs build on related theoretical foundations and emphasize the importance of experience and reflection, no matter the beholders of the beliefs, may it be children or adults, students, or teachers. For the same reason, the methodological approach to assessing development in beliefs through a qualitative, longitudinal DBR study, which includes triangulation of data types, may also be considered transferable to other contexts.

10.5.3. Importance

In continuation of the generality, it is relevant to discuss the importance of the study and the potential impact of its contributions.

Theoretically, the study contributes to an understanding of the character of middle school students' beliefs about mathematics as a discipline in a Danish context. In addition to supporting existing research, these findings give a detailed characteristic of their beliefs in three aspects of mathematics, namely the three forms of overview and judgment. They furthermore relate the students' beliefs to the element of reflection as well as attitude, for instance, by showing the students' associations between mathematics and negative emotions. As the character of students' beliefs reflect their experience in the classroom, the findings also constitute a highly relevant contribution to the discussion raised in chapter 1 concerning approaches to mathematics teaching, and whether the teaching supports the development of desired beliefs within an educational system that emphasizes mathematical competence and *Allgemeinbildung* in a democratic society.

Another theoretical contribution concerns the process of developing students' beliefs. The findings provide information on the stability of students' beliefs, especially related to their centrality and the influence of attitudes on this stability. The study contributes to a research field with only a few investigations of the development of students' beliefs in a longitudinal perspective.

Methodologically, the study offers a suggestion for approaches towards changing students' beliefs as well as methods for accessing them and measuring any potential development. In light of existing research, the combination of a longitudinal DBR intervention and triangulation of data is rare, and the findings of this study indicate that this method has something to offer in terms of understanding the development of beliefs. Existing research as well as this study clearly show that students' beliefs have an essential influence on their learning, their motivation, and their engagement in the classroom. In order to ensure that the students develop beliefs that may increase these factors, it is important to understand how the best opportunities for this development are attained, and what might be the challenges and obstacles in this process. In relation to the DBR approach, it is furthermore noticeable that the collaboration between researcher and teachers was indeed beneficial in terms of professional and didactical development. This was highlighted by the teachers as one of the most important sources for their personal and professional outcome of the intervention.

From a pragmatic perspective, one of the contributions of the study is that it may raise awareness of the importance of focusing on the students' beliefs about mathematics as a discipline, thus increasing teachers' attentiveness to these as well as to the relation between their didactical choices and the students' perception of and approach to mathematics. In addition, it might emphasize the potential of including aspects of mathematics that are overlooked in the teaching, such as the historical development of mathematics or philosophical elements of the subject. The teaching principles constitute a contribution in themselves as a design proposal for a competence-oriented teaching approach, which includes the students' development of mathematical overview and judgment.

10.6. Implications for practice

It is not particularly surprising that the middle school students participating in this study largely associate mathematics with the school subject, and that their associations with the school subject primarily are related to a dualistic perspective on mathematics with a clear emphasis on numbers, arithmetic, rules, methods, results, and speed. This finding is in line with previous research. As mentioned earlier, it can largely be said to reflect which experiences the students have had in the classroom in their schooling so far. Judging from the data collected in this study, these experiences have not focused on aspects related to mathematics as a discipline—or at least not in a way that has made an impact on the students' beliefs about mathematics. This is also not surprising. As described in section 2.3.3, mathematics as a discipline—represented by the three forms of overview and judgment—does not seem to be prioritized in the mathematics teaching in grade 1-9 in Denmark.

However, the Danish competencies framework, which has been part of the curriculum since 2009 (and thereby during all the school years of the students in the study), certainly does not invite to a dualistic perspective on mathematics. A competence-oriented teaching approach should thus ideally entail a more relativistic perspective on mathematics. One might even argue that a large part of the relativistic perspective lies in the development of overview and judgment. In section 2.2.3, the relativistic perspective is described as viewing mathematics as a coherent and dynamic system of knowledge and a useful endeavor with a focus on sense-making, logical thought, and understanding (e.g., Grouws, 1996; Oaks, 1989; Schoenfeld, 1992). Where OJ1 supports the sense of usefulness, OJ2 contributes to an insight into the dynamic nature of mathematics as well as the coherence between mathematical content areas. This can also be said about OJ3, which further increases the focus on sense-making etc. Hence, when the part of the competencies framework constituted by overview and judgment is not prioritized in mathematics education from the beginning of their school years, the students cannot be expected to develop a relativistic perspective on mathematics.

As found in a large part of the literature concerning students' beliefs about mathematics as a discipline, the relativistic perspective on mathematics is considered advantageous in at least two regards. It can be empirically related to a higher performance level (e.g., Gattermann et al., 2012; Grigutsch, 1998; Grouws, 1996), and it increases the motivation and self-concept of the students (Grigutsch, 1998). Furthermore, as argued by Ernest (2015), aspects related to a relativistic perspective of mathematics may potentially increase the students' appreciation of mathematics and provide them with social empowerment. Following this argumentation, the development of students' mathematical overview and judgment should begin in the early school years. Yet, as seen in this study, increasing the focus on the students' overview and judgment can be a complicated and challenging matter, which entails a great effort from especially the teachers. Not only is the teaching approach essential, the teachers' own knowledge, beliefs and reflection play important roles in creating a nuanced and broad perspective on mathematics as a discipline. This study presents preliminary suggestions of how this task may be approached in relation to general principles for the teaching and the process of developing students' beliefs. However, it does not, as mentioned, offer a guide or procedure. Instead, the study may spark awareness and reflection on the nature of mathematics, why we teach it, and how we want students to perceive, approach, and apply mathematics. Perhaps these general and philosophical, yet essential questions may form the starting point of didactical dialogues among teachers.

In line with earlier sections of this discussion, these considerations of the study's implications for practice initiate further questions:

- The teachers in this study emphasized the importance and benefits of regular didactical discussions. What initiatives may be taken to facilitate and prioritize discussions in teachers' work?
- One of the main challenges connected to developing students' beliefs in this study was connected to implementing the element of reflection. How can teachers become familiar with this crucial element in such a way that they feel equipped to incorporate it into their teaching?
- The Danish mathematics curriculum suggests that the development of students' mathematical overview and judgment is addressed throughout the compulsory grades, but at the latest from lower secondary school. In light of this study, it seems reasonable to begin this development earlier—perhaps even from the very earliest school years. But what might such an implementation entail, and how can it be approached? Are certain elements of overview and judgment more relevant or suited for young students? And what would such an initiative require of the teachers as well as the students?

Chapter 11: Conclusions

This chapter concludes the study by summarizing the key research findings in relation to the research aims and questions, and by discussing the value and contribution hereof. Furthermore, it reviews the limitations of the study and proposes opportunities for future research.

11.1. Answering the research questions

As stated in the introduction of this dissertation, the study's overall aim was to develop beliefs that contribute to the students' mathematical competence as part of their *Allgemeinbildung* and democratic citizenship. To reach this aim, the focus of the intervention was on increasing their insight into the nature and role of mathematics through the notion of overview and judgment—an already existing part of the Danish mathematics curriculum. The hypothesis that *a longitudinal change of focus in the teaching of mathematics can contribute to a change in middle school students' beliefs about mathematics – specifically that an increased focus on mathematical overview and judgment can influence their beliefs about mathematics as a discipline*, can to some extent be confirmed, though only in regard to the level of reflection and within certain aspects. Part of the argumentation for the research design was that an early implementation of overview and judgment could counteract the increasing stability of the students' non-evidentially held beliefs. However, the findings may give reason to argue that some of the students' beliefs, particularly those based on their experiences in the classroom, already have a high degree of stability. Hence, it becomes proportionally difficult to change these beliefs. For example, the students' beliefs about the nature of mathematics as a subject area, in the study, largely coincided with their beliefs about school mathematics. Thereby, the students' insights into the nature of mathematics becomes restricted to the nature of the school subject, and, because of their stability, this may be difficult to change. The aim of contributing to the students' perception of the relevance of mathematics and its role outside a school context was therefore not reached entirely. Still, the findings indicate that a continued intervention might have offered potential for approaching this aim. As the findings related to the research questions have been elaborated in the subdiscussions and summarized in the main discussion, I shall, here, attempt to give concise answers to each of the questions based on the findings.

Research question 1: What characterizes Danish middle school students' beliefs about mathematics as a discipline?

The findings of this study indicate that Danish middle school students possess beliefs about mathematics as a discipline, which do not in essence differ from beliefs held by students in other countries. Although they find mathematics important to learn, their beliefs about the discipline are either non-existent or equal to their beliefs about mathematics as a school subject, with a strong tendency towards a dualistic perspective of mathematics. The students' beliefs about mathematics as a discipline are generally characterized by a low level of reflection. Beliefs developed from experiences in the classroom appear central and stable, whereas non-existent or derived beliefs (especially about the historical development of mathematics) appear peripheral and thus easier to develop and modify.

Research question 2: Which changes can be detected in the students' beliefs about mathematics as a discipline after a longitudinal intervention that focuses on developing the students' mathematical overview and judgment?
The design of the intervention was overall based on the three forms of overview and judgment in combination with a changing process, defined in three steps: awareness, experience, and reflection. After two years of this intervention, the students' beliefs about mathematics as a discipline primarily seem to have developed in terms of their level of reflection, and in terms of exemplification, justification, and consistency. Furthermore, the students have become more aware of aspects connected to mathematics as a discipline, which constitutes a first step towards changes in their beliefs. This development can be linked to the intervention, as the students refer to activities and examples therefrom. However, the content of their beliefs did not change in essence; nor did the predominantly dualistic perspective of mathematics as a discipline.

11.2. Main contributions

The main contributions of the study can be divided into theoretical, pragmatic, and methodological contributions.

11.2.1. Theoretical contributions

Theoretically, the study offers a so far unexplored understanding of the character of Danish middle school students' beliefs about mathematics as a discipline, indicating that it resembles the character of students' beliefs investigated in international studies. Given that different countries' curricula, teaching traditions, and educational cultures are not equivalent, this contribution may give reason for speculation and investigation of explanation of this similarity in the students' beliefs.

The study, furthermore, contributes to an understanding of the processes involved in developing students' beliefs about mathematics (as a discipline), enabled by the longitudinal design. The findings show that beliefs that have already been established and confirmed through experiences in the classroom are notably more difficult to change than beliefs about aspects, of which the students have not yet become aware or had the opportunity to reflect on, partly because these beliefs are established instead of changed. In addition, the findings suggest that an articulation of beliefs leads to an increased awareness, which is necessary for the students' possible reflection on any potential cognitive conflicts. Moreover, the study shows the importance of *reflection* in the process of developing beliefs, as there are examples of students not completing a development in their beliefs, most likely due to a low level of reflection, which was possibly caused by challenges connected to the implementation of the intervention.

11.2.2. Methodological contributions

The study's methodological contributions mainly concern how beliefs are accessed, developed, and, not least, how a development in beliefs can be assessed. The complexity and inaccessibility of beliefs make an assessment difficult. Yet, this study illustrates how these difficulties to some extent can be accommodated through triangulation of data types, making it possible to compare beliefs related to different contexts and sources of activation. Furthermore, the longitudinal design of the study enabled a chronological comparison between time phases and data types, thus making it possible to infer a potential development in the students' beliefs.

In addition, the DBR approach has contributed to accessing the students' beliefs, as it enables opportunities to, at the same time, study and develop them in the classroom context where they are formed. The experiences connected to the data collection might contribute to an increased understanding of influential factors in accessing beliefs, as for example the importance of *activation.*

The frameworks used for developing the students' beliefs (awareness, experience, and reflection) as well as assessing them (levels of exemplification, justification, and consistency), support the above-stated essentiality of the reflection. Thereby, the study contributes with a coherent strategy for developing beliefs and assessing this development.

11.2.3. Pragmatic contributions

In practice, this study offers suggestions, recommendations and ideas for a teaching approach seeking to develop students' overview and judgment as part of their mathematical competence. Although the pragmatic outcome, represented by the design teaching principles, does not offer a fixed and finished guide for teaching, it may still contribute to an increased awareness of not only the overlooked notion of mathematical overview and judgment, but also of the importance of students' beliefs about mathematics as a discipline. This dimension of the students' mathematics-related belief system influences the students' beliefs about other belief dimensions, and thereby their learning of (and approach to) mathematics. Conversely, the belief dimensions related to a school setting influence the students' beliefs about mathematics as a discipline. As this dimension so to speak constitutes the "link" to the insight into the nature and role of mathematics in the world, the experiences in the classroom become of immense importance in making the students see the relevance of mathematics, and eventually to contribute to their *Allgemeinbildung* and the democratic citizenship in regard to mathematical competence.

Although not initially a part of its focus, the study illustrates the challenges connected to the teachers' role in the process of developing students' beliefs. Their beliefs, the design, and the goals of the intervention, as well as pragmatic circumstances may all influence this process. Based on elements of the discussion in this study, there may thus be incentives for further investigations in this regard.

11.3. Limitations and weaknesses

Overall, the study is conducted under the restrictions determined by the nature of beliefs. They cannot be directly measured or observed. They must be accessed through interpretive processes. Thereby, the results of the analyses are to some extent subjective and dependent on the context in which the beliefs have been approached. This, of course, affects the trustworthiness of the findings, which may offer strong indications and depict tendencies, however connected to some degree of uncertainty. Particularly results that build solely on one type of data (as for example the analyses of the two questionnaires), should be perceived as a snapshot of the students' beliefs, which may not be similar in another context or at another time.

The findings indicate only minor signs of development in the students' beliefs and, although these to some extent can be linked to the intervention, the possibility that they may be caused by other factors cannot be ruled out. For example, the increased level of reflection may be related to the students' progressed age and maturity, and the higher level of exemplification may possibly be caused by experiences outside school.

The unique circumstances caused by the COVID-19 pandemic may also have affected the data collection as well as the possibility for implementing the intervention. Furthermore, it may have influenced the students' attitudes, emotions, and motivation, which eventually may possibly have affected the students' beliefs.

However, this dissertation offers thick descriptions of all steps and decisions taken in the study, which contribute to its trustworthiness as well as the generality. Furthermore, attempts have been made to accommodate the limitations and weaknesses through the longitudinal design, the triangulation of data, and regular feedback from peers.

11.4. Recommendations

Based on the overall study, I wish to present my main recommendations for practice as well as for future research.

First, the study is a clear indication of the importance of an early focus on the development of students' beliefs about mathematics as a discipline. Experiences in the classroom play an essential part in this development, and the early formed beliefs often become central and robust, and thus subsequently determine the students' perception of future experiences. Hence, it is crucial that teachers, policy makers and others who may influence these beliefs become conscious of their potential influence on them, and use this influence determinedly to contribute to the development of what may be considered desirable beliefs. In a Danish context, I recommend an increased implementation of the notion of mathematical overview and judgment, preferably from the early school years.

Second, following the first recommendation, young students' beliefs about mathematics as a discipline may indeed be of interest in the field of mathematics education. Although the methodological implications and challenges are probably even more pronounced than what was the case among the students in this study, findings here suggest that the importance of early formed beliefs should be taken seriously and thoroughly investigated. Such investigations may thereby contribute to an increased understanding of how and when students' beliefs are formed, and which processes are involved. Furthermore, methodological experiences may be gained in relation to accessing young students' beliefs.

Third, this study offers a basis for discussing desirable, favorable, or ideal beliefs about mathematics as a discipline. For the first recommendation to be followed, such a discussion is central. It includes essential considerations connected to the justification of mathematics education and the desired learning outcome for students on all educational levels.

Finally, the element of reflection seems to be of central value for the development of students' beliefs. At the same time, this element was, in fact, what caused the teachers in this study the most trouble in terms of implementation. Therefore, I suggest that the matter of including reflection in the teaching of mathematics as a didactical tool and as part of a competence-oriented approach to learning becomes an object of research in mathematics education, as well as education in general.

11.5. Closing remarks

If students' beliefs about mathematics as a discipline are to be changed in the future, an understanding of the processes and challenges involved in the *development* of beliefs is essential. Considering and addressing the development of students' beliefs about mathematics as a discipline is very much a matter of taking a step back and looking at mathematics education from a broader perspective. My personal experience, which is supported by the findings of this study, is that this step is rarely taken in the classroom, in professional discussions among teachers, or in the offices of policy makers. It is my hope that this dissertation may be a contribution to an increased and fruitful dialogue about this matter.

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June 2019

Contract between *participating school* and the Ph.D. project "Middle school students' beliefs about mathematics as a discipline".

The project examines middle school students' beliefs about mathematics, as well as which teaching initiatives can support a positive development in this area. The purpose of the project is an increased academic and educational benefit for all students.

The project's intervention in two 6th grade classes proceeds over two school years starting in August 2019. However, there will be initial meetings between the teachers and the Ph.D. student in June 2019. The intervention will end in June 2021. During the project, data on students' names, class, gender, learning behavior, and beliefs about mathematics will be collected. All data will be anonymized so that no persons can be identified, and all data will be stored and treated confidentially in accordance with the code of conduct for research integrity.

The intervention is based on principles for the teaching designed by the Ph.D. student and the participating teachers, on the basis of which the teachers plan the teaching. These principles are continuously adjusted. There is thus a high degree of teacher involvement in the project, which presupposes that, as far as possible, the two teachers remain affiliated in the classes throughout the project. The lessons are observed at 2-3 week intervals, preferably during double lessons.

Project manager:	Maria Kirstine Østergaard, Ph.D. student at AU / UCC
Data manager:	Aarhus University Nordre Ringgade 1 8000 Aarhus dpo@au.dk
School:	[redacted]
School principal:	[redacted]
Participating teachers:	[redacted]

The project covers a total of 75 hours per. teacher with DKK [redacted] per hour. Payments transferred to the school via invoice:

[redacted]

DPU, Aarhus University
Tuborgvej 164
2400 København NV

[redacted]

[redacted]

The hours are distributed as follows:

2019/2020: 40 hours, divided into

- 10 h. for preparation in June / August 2019
- 5 h. for evaluation in Dec. 2019
- 5 h. for evaluation in Jun. 2020
- 20 h. for regular meetings throughout the school year, 1-2 times a month

2020/2021: 35 hours, divided into

- 5 h. for preparation in Aug. 2020
- 5 h. for evaluation in Dec. 2020
- 5 t. h. for final evaluation in Jun. 2021
- 20 h. for regular meetings throughout the school year, 1-2 times a month

The school undertakes to be involved in the project with the two classes for two school years (2019/2020 and 2020/2021) as well as to accommodate the project in the participating teachers' portfolios.

Teacher

______ ______________________

Date *Signature*

Teacher

______ ______________________

Date *Signature*

Principal

______ ______________________

Date *Signature*

Ph.D. student

______ ______________________

Date *Signature*

Questions in questionnaire*

Students' beliefs about mathematics as a discipline

1.	What do you think of when you hear the word "mathematics"?
2.	How much do you like math in school? 😩 🙁 🙂 😀
3.	Do you think you are good at math? 😩 🙁 🙂 😀
4.	What do you like the most about math?
5.	What do you like the least about math?
A1.	Do you think it is important for you to learn math? Yes, because: / No, because:
A2.	Do you think it is important for everybody in Denmark to learn math? Yes, because: / No, because:
A3.	What do you experience and think that math is used for in your daily life?
A4.	What do you experience and think that math is used for in society?
A5.	Do you think that mathematics is more important now than 100 years ago? Why / why not?
B1.	How do you imagine that the math, you learn at school, has come into being?
B2.	Why do you think somebody came up with mathematics?
B3.	When do you think mathematics came in to being? (Specify year or period)
B4.	*2019:* Did you ever learn anything in school about how mathematics came in to being? No / Yes. What was it about? *2021:* What did you learn in school about the history of mathematics?
C1.	*2019:* What is the difference between math and other school subject? *2021:* What do you think is the difference between having math and other subjects?
C2.**	What is the most important in math? Choose max. 3 things: a. To get a correct result b. To be able to remember *(in 2021:* To know rules and formulas by heart*)* c. To use the correct methods d. To come up with your own methods for solution *(only in 2019)* e. To understand what the teacher means *(in 2021: … what the teacher* explains*)* f. To be able to explain what you think g. To solve problems h. To know the times tables i. To be able to find patterns j. To get good ideas k. Other: ____________________ l. To be able to make decisions *(only in 2021)*
C3.	What is a mathematical problem?
C4.	What do you think a mathematician does?
D1.	Draw yourself working with mathematics that you find fun:
D2.	Explain your drawing:
E1.	What do you remember best from the math lessons in 6th and 7th grade? *(only in 2021)*
E2.	When do you think you have learned most, and why? *(only in 2021)*
E3.	What have been the funniest during math class during the last two years? *(only in 2021)*

** The original 2019-questionnaire is in a physical format and includes space for writing and drawing, with two or three question on each page (size A4). The 2021-questionnaire is digital, though with a physical page for questions D1 and D2.*

*** In the 2021-questionnaire, the options in question C2 are presented in random order.*

Coding protocol incl. distribution of answers

2019: N=43 (Class X:21, Class Y: 22)
2021: N=38 (Class X: 19, Class Y: 19)

Numbers in parenthesis show results from 2019.

Question	Code	1	Number of st. from class X	Number of st. from class Y	2	X	Y	3	X	Y	4	X	Y	0	X	Y
1. What do you think of when you hear the word "mathematics"?		Emotionally negative ("it is difficult", "I cannot do it", "oh God, not that again", "boo", "oh no"	6 (5)	3 (7)	Content, specific ("multiplication", "fractions")	7 (8)	9 (4)	Content, general ("numbers", "calculating", "my teacher", "calculations")	6 (9)	7 (10)	Emotionally positive ("fun", "challenging")	1 (1)	3 (4)	No answer / misunderstood ("nothing")	0 (2)	0 (0)
2. How much do you like math in school?			2 (4)	0 (1)		7 (3 (2,5: 3))	3 (4 (2,5: 3))		8 (10)	9 (9)		2 (1)	7 (5)	No answer	0 (0)	0 (0)
3. Do you think you are good at math?			0 (3)	0 (3)		8 (4 (2,5: 2))	3 (3)		6 (11)	14 (10)		5 (1)	2 (6)	No answer	0 (0)	0 (0)
4. What do you like the most about math?		Don't know	0 (3)	0 (0)	Emotionally negative ("nothing")	4 (4)	0 (3)	General ("homework", "calculating")	2 (3)	3 (1)	Specific content ("division", "fractions", "math games")	14 (9)	16 (16)	No answer	0 (2)	0 (1)
5. What do you like the least about math?		Don't know	1 (1)	1 (1)	Emotionally negative ("everything")	1 (4)	2 (4)	General ("homework", "calculating")	1 (4)	0 (1)	Specific content ("division", "fractions", "math games")	16 (10)	16 (15)	No answer	0 (2)	0 (1)
A1. Do you think it is important for you to learn math? Yes, because: / No, because:		No	1 (2)	1 (1)	Yes Unspecified justification or vague ("it is important", "you need it")	1 (2)	0 (6)	Yes Generally specified ("to get an education", "to get a job", "you	17 (16)	18 (11)	Yes Further specified ("I'm going to be an engineer")	0 (1)	0 (4)	No answer	0 (0)	0 (0)

							use it for everything")								
A2. Do you think it is important for everybody in Denmark to learn math? Yes, because: / No, because:	No	5 (3)	4 (3)	Yes Unspecified justification or vaque ("it is important", "you need it")	1 (1)	3 (7)	Yes Generally specified ("to get an education", "to get a job", "you use it for everything")	12 (15)	11 (10)	Yes Further specified ("I'm going to be an engineer")	1 (2)	1 (2)	No answer	0 (0)	0 (0)
A3. What do you experience and think that math is used for in your daily life?	Nothing / don't know	1 (1)	0 (2)	Not specified ("everything")	0 (1)	0 (2)	Shopping + possibly something else	10 (13)	11 (13)	Several things	8 (6)	7 (5)	No answer / misunderstood	0 (0)	0 (0)
A4. What do you experience and think that math is used for in society?	Nothing / don't know	0 (7)	0 (4)	Everyday things ("Shopping")	4 (5)	5 (6)	General, society ("everything", "work", "money")	7 (7)	5 (5)	Specific, society ("to build roads", "to pay tax")	8 (2)	11 (6)	No answer / misunderstood	0 (0)	0 (1)
A5. Do you think that mathematics is more important now than 100 years ago? Why / why not?	More important 100 years ago OR equally important, no or vague justification	0 (6)	2 (6)	More important now, no or vague justification	6 (2)	3 (1)	More important 100 years ago with justification	3 (0)	6 (6)	More important now with justification	7 (8)	7 (10)	No answer / misunderstood / don't know	0 (5)	1 (1)
B1. <u>How</u> do you imagine that the math you learn at school has come into being?	Don't know or deficient response	2 (6)	2 (7)	Emotionally negative	0 (2)	0 (0)	Vague justification ("somebody just thought of it", "to measure something")	4 (8)	10 (7)	Reflected response	3 (3)	5 (4)	No answer / misunderstood	10 (2)	2 (4)
B2. <u>Why</u> do you think somebody came up with mathematics?	Don't know	2 (3)	3 (0)	Emotionally negative	0 (4)	0 (1)	Vague justification	13 (9)	11 (12)	Reflected response	4 (4)	5 (5)	No answer / misunderstood	0 (1)	0 (4)
B3. <u>When</u> do you think mathematics came in to being? (specify year or period)	Don't know	5 (1)	0 (3)	Specific (random) year ("1773", "1900")	4 (14)	11 (14)	Period (random) ("the Iron Age")	9 (3)	3 (4)	Justified time indication ("since humans came into being")	2 (1)	4 (1)	No answer / misunderstood	0 (2)	1 (0)
B4. 2019: Did you ever learn anything in school about how mathematics came in to being? No /	2019: No 2021: Nothing	0 (9)	0 (18)	2019: Yes, no specification 2021: Don't remember	4 (0)	2 (0)	2019: Yes, vague specification 2021: Vague specification	8 (2)	8 (1)	2019: Yes, precise specification 2021: Precise specification	6 (7)	8 (3)	No answer / misunderstood	1 (2)	1 (0)

Yes. What was it about? 2021: What did you learn in school about the history of mathematics?															
C1. What is the difference between math and other school subject?	Don't know / no difference / deficient response	3 (5)	1 (2)	Emotional response (”I don't like math”, ”I like English better”)	6 (7)	8 (5)	Content (”There are more numbers in math”, ”in math you calculate”)	7 (9)	8 (12)	Working methods (”we work more with computer”, ”there is more group work”)	4 (1)	4 (4)	No answer / misunderstood	0 (0)	0 (0)
C2. What is the most important in math? Choose max. 3 things:															
C2a. To get a correct result	Marked	5 (9)	4 (6)												
C2b. To know rules and formulas by heart		9 (7)	6 (1)												
C2c. To use the correct methods		8 (5)	7 (7)												
C2d. To come up with your own methods for solution (2019)		- (10)	- (12)												
C2e. To understand what the teacher explains		8 (9)	4 (8)												
C2f. To be able to explain what you think		8 (6)	12 (10)												
C2g. To solve problems		4 (4)	7 (2)												
C2h. To know the times tables		3 (8)	2 (6)												
C2i. To be able to find patterns		2 (0)	1 (1)												
C2j. To get good ideas		2 (2)	5 (6)												
C2k. Other:___		1 (1)	1 (3)												
C2l To be able to make decisions (2021)		1 (-)	7 (-)												

C3. What is a mathematical problem?	Don’t know	1 (14)	4 (13)	Vague (”a problem with math”)	7 (3)	8 (7)	Backwards understanding (”when you calculate wrong”)	0 (0)	0 (0)	Sign of understanding	10 (1)	7 (1)	No answer / misunderstood	0 (3)	0 (1)
C4. What do you think a mathematician does?	Don’t know	1 (1)	2 (4)	Wrong or irrelevant response.	3 (1)	2 (1)	Vague response (”does math”, ”calculates”)	8 (12)	8 (12)	Sign of understanding (”works with mathematical problems”)	7 (5)	7 (3)	No answer / misunderstood	0 (2)	0 (2)
¤1. What do you remember best from the mathematics classes in 6th and 7th grade? (2021)	Don’t know / can’t remember / nothing	2	2	Not specified / not related to mathematics	5	0	Specified, not related to intervention	2	8	Specified, related to intervention	10	9	No answer / misunderstood	0	0
E2. When do you think you learned most, and why? (2021)	Don’t know / can’t remember / nothing	2	0	Not specified / not related to mathematics	10	6	Specified, not related to intervention	5	8	Specified, related to intervention	2	5	No answer / misunderstood	0	0
E3. What has been the most fun in mathematics class in the last two years? (2021)	Don’t know / can’t remember / nothing	4	2	Not specified / not related to mathematics	5	0	Specified, not related to intervention	6	8	Specified, related to intervention	4	9	No answer / misunderstood	0	0
D1. Draw yourself working with mathematics that you find fun:	Emotionally negative	1 (2)	0 (0)	Traditional student role (alone, sitting by a desk, solving tasks)	7 (8)	4 (14)	Other, but in a school setting	6 (6)	9 (5)	Non-school setting	4 (3)	1 (2)	No answer / misunderstood	2 (2)	6 (1)
D2 Explain your drawing	Emotionally negative	1 (2)	0 (0)	Traditional student role (alone, sitting by a desk, solving tasks)	7 (8)	4 (14)	Other, but in a school setting	6 (6)	9 (5)	Non-school setting	4 (3)	1 (2)	No answer / misunderstood	2 (2)	6 (1)

Appendix D: Distribution of answers + average code value for each question

Questionnaire 2021.

Number of answers in categories 1, 2, 3, 4 and 0.

Class X: N=21 (2019), N=19 (2021)

Class Y: N=22 (2019), N=19 (2021)

		Code 1						Code 2						Code 3						Code 4						Code 0						Average code value						Diff. in av. code value		
		Code 1 2019			Code 1 2021			Code 2 2019			Code 2 2021			Code 3 2019			Code 3 2021			Code 4 2019			Code 4 2021			Code 0 2019			Code 0 2021			2019			2021			2019-2021		
	Question/cat.	1X	1Y	Tota	1X	1Y	Tota	2X	2Y	Tota	2X	2Y	Tota	3X	3Y	Tota	3X	3Y	Tota	4X	4Y	Tota	4X	4Y	Tota	0X	0Y	Tota	0X	0Y	Tota	X 19	Y 19	Av. 19	X 21	Y 21	Av. 21	Diff. X	Diff. Y	Diff. all
1.	What do you think of, when you hear the word ”mathe	5	7	12	6	3	9	8	4	12	7	9	16	9	10	19	6	7	13	1	4	5	1	3	4	2	0	2	0	0	0	2,08	2,44	2,26	2,10	2,45	2,29	0,02	0,01	0,03
2.	How much do you like math in school?	4	1	5	2	0	2	6	7	13	7	3	10	13	12	25	8	9	17	1	5	6	2	7	9	0	0	0	0	0	0	2,46	2,84	2,65	2,53	3,21	2,87	0,07	0,37	0,22
3.	Do you think you are good at math?	3	3	6	0	0	0	6	3	9	8	3	11	13	10	23	6	14	20	1	6	7	5	2	7	0	0	0	0	0	0	2,52	2,86	2,69	2,84	2,95	2,89	0,32	0,08	0,21
4.	What do you like the most about math?	3	0	3	0	0	0	4	3	7	4	0	4	3	1	4	2	3	5	9	16	25	14	16	30	2	1	3	0	0	0	2,67	3,48	3,07	3,50	3,84	3,67	0,83	0,37	0,60
5.	What do you like the least about math?	1	1	2	1	1	2	4	4	8	1	2	3	4	1	5	1	0	1	10	15	25	16	16	32	2	1	3	0	0	0	2,90	3,27	3,09	3,68	3,63	3,66	0,78	0,36	0,56
	Attitude of mathematics (overall, q. 1-5)																															2,53	2,98	2,75	2,93	3,22	3,07	0,40	0,24	0,32
A1.	Do you think it is important for you to learn math?	2	1	3	1	1	2	2	6	8	1	0	1	16	11	27	17	18	35	1	4	5	0	0	0	0	0	0	0	0	0	2,76	2,82	2,79	2,84	2,89	2,87	0,08	0,08	0,08
A2.	Do you think it is important for everybody in Denmark	3	3	6	5	4	9	1	7	8	1	3	4	15	10	25	12	11	23	2	2	4	1	1	2	0	0	0	0	0	0	2,76	2,50	2,63	2,47	2,47	2,47	-0,29	-0,03	-0,15
A3.	What do you experience and think that math is used fo	1	2	3	1	0	1	1	2	3	0	0	0	13	13	26	10	11	21	6	5	11	8	7	15	0	0	0	0	0	0	3,14	2,95	3,05	3,32	3,39	3,35	0,17	0,43	0,30
A4.	What do you experience and think that math is used fo	7	4	11	0	0	0	5	6	11	4	5	9	7	5	12	7	5	12	2	6	8	8	11	19	0	1	1	0	0	0	2,19	2,50	2,35	3,21	3,29	3,25	1,02	0,79	0,90
A5.	Do you think that mathematics is more important now	6	6	12	0	2	2	2	1	3	6	3	9	0	6	6	3	6	9	8	10	18	7	7	14	5	1	6	0	1	1	2,00	2,75	2,40	3,06	2,84	2,94	1,06	0,09	0,54
	Application of mathematics (overall, q. A1-A5)																															2,57	2,70	2,64	2,98	2,98	2,98	0,41	0,27	0,33
B1.	How do you imagine that the math, you learn at school	6	7	13	2	2	4	2	0	2	0	0	0	8	7	15	4	10	14	3	4	7	3	5	8	2	4	6	10	2	12	2,19	2,00	2,09	1,37	2,74	2,05	-0,82	0,74	-0,04
B2.	Why do you think somebody came up with mathemati	3	0	3	2	3	5	4	1	5	0	0	0	9	12	21	13	11	24	4	5	9	4	5	9	1	4	5	0	0	0	2,57	2,64	2,60	3,00	2,95	2,97	0,43	0,31	0,37
B3.	When do you think mathematics came in to being? (sp	1	3	4	5	0	5	14	14	28	4	11	15	3	4	7	9	3	12	1	1	2	2	4	6	2	0	2	0	1	1	2,00	2,14	2,07	2,40	2,47	2,44	0,40	0,34	0,37
B4.	What did you learn in school about the history of math	9	18	27	0	0	0	0	0	0	4	2	6	2	1	3	8	8	16	7	3	10	6	8	14	2	0	2	1	1	2	2,15	1,50	1,81	2,95	3,16	3,05	0,80	1,66	1,24
	Historical development of mathematics (overall, q.B1-B4)																															2,23	2,07	2,14	2,43	2,83	2,63	0,20	0,76	0,48
C1.	What is the difference between math and other school	5	2	7	3	1	4	7	5	12	6	8	14	9	12	21	7	8	15	1	4	5	4	4	5	0	0	0	0	0	0	2,27	2,78	2,53	2,60	2,71	2,55	0,33	-0,07	0,02
C2.	What is the most important in math? Choose max. 3 things:																																							0,00
C2a.	*To get a correct result*	9	6	15	5	4	9																			12	16	27	14	15	29									0,00
C2b.	*To be able to remember*	7	1	8	9	6	15																			14	21	35	10	13	23									0,00
C2c.	*To use the correct methods*	5	7	12	8	7	15																			16	15	31	11	12	23									0,00
C2d.	*To come up with your own methods for solution (only 2*	10	12	22																						11	10	21												0,00
C2e.	*To understand what the teacher means/explains*	9	8	17	8	4	12																			12	14	26	11	15	26									0,00
C2f.	*To be able to explain what you think*	6	10	16	8	12	20																			15	12	27	11	7	18									0,00
C2g.	*To solve problems*	4	2	6	4	7	11																			17	20	37	15	12	27									0,00
C2h.	*To know the multiplication tables*	8	6	14	3	2	5																			13	16	29	16	17	33									0,00
C2i.	*To be able to find patterns*	0	1	1	2	1	3																			21	21	42	17	18	35									0,00
C2j.	*To get good ideas*	2	6	8	2	5	7																			19	16	35	17	14	31									0,00
C2k.	*Other*	1	3	4	1	1	2																			20	19	39	18	18	36									0,00
C2l.	*To be able to make decisions (only 2021)*				1	7	8																						18	12	30									0,00
C3.	What is a mathematical problem?	14	13	27	1	4	5	3	7	10	7	8	15	0	0	0	0	0	0	1	1	2	10	7	17	3	1	4	0	0	0	1,14	1,41	1,28	3,06	2,53	2,78	1,91	1,12	1,50
C4.	What do you think a mathematician does?	1	4	5	1	2	3	1	1	2	3	2	5	12	12	24	8	8	16	5	3	8	7	7	14	2	2	2	0	0	0	2,81	2,45	2,76	3,11	3,05	3,08	0,30	0,60	0,32
	Nature of mathematics (overall, q. C1+C3+C4)																															2,08	2,22	2,19	2,92	2,76	2,81	0,85	0,55	0,62
D1.	Draw yourself working with mathematics that you find	2	0	2	1	0	1	8	14	22	7	4	12	6	5	11	6	9	15	3	2	5	4	1	5	2	1	3	2	6	8	2,29	2,32	2,30	2,45	1,95	2,20	0,16	-0,37	-0,11
D2.	Explain your drawing	2	0	2	1	0	1	8	14	22	7	4	12	6	5	11	6	9	15	3	2	5	4	1	5	2	1	3	2	6	8	2,29	2,32	2,30	2,45	1,95	2,20	0,16	-0,37	-0,11
	Mathematical activity (overall, q. D1-D2)																															2,29	2,32	2,30	2,45	1,95	2,20	0,16	-0,37	-0,11
E1.	What do you remember best from the math lessons in	-	-	-	2	2	4	-	-	-	5	0	5	-	-	-	2	8	10	-	-	-	10	9	19	-	-	-	0	0	0	-	-	-	3,05	3,26	3,16	-	-	-
E2.	When do you think you have learned most, and why?	-	-	-	2	0	2	-	-	-	10	6	16	-	-	-	5	8	13	-	-	-	2	5	7	-	-	-	0	0	0	-	-	-	2,37	2,95	2,66	-	-	-
E3.	What have been the funniest during math class during	-	-	-	4	2	6	-	-	-	5	0	5	-	-	-	6	8	14	-	-	-	4	9	13	-	-	-	0	0	0	-	-	-	2,53	3,26	2,89	-	-	-
	Evaluation of intervention (overall, q. E1-E3)																																		2,65	3,16	2,90			

Interview guide, students 2019

Name: Class: Date:

Intro: Thanks for letting me talk to you. I'm recording our conversation, but I am the only one who will listen to the recording.

We are going to talk a little bit about mathematics and what you think about it. Some of the questions are a bit reminiscent of those that were in the questionnaire and some of them are different. It was very interesting to read your answers in the questionnaire and I would love to hear a little more about what you think.

It is important that you just say exactly what you are thinking. My job is to try to make mathematics teaching as good as possible and for that, I need to know as much as possible about how students are actually thinking. So, the more honest you are the better. And neither your teacher nor your parents will know what we're talking about unless you tell them yourself.

Are you ready? Then we begin.

Research questions	Interview questions	Follow-up	Individual notes
General attitude towards math			
What are the student's feelings and attitudes towards mathematics?	Tell me about how you feel about math. Have you always felt that way with math, or was it different when you were younger? Is the mathematics lessons different at this school than at the old one?	What do you like the best to do in math?	
Academic preferences	What is your favorite subject?	If not math: What is it that you like about that subject?	
Learning behavior			
What is the student's perception of own learning behavior?	How do you work in math? *(refer to drawing in questionnaire)*	How do you work individually? How do you think it is to work with you in a group?	

		Are you active in class? Do you participate in classroom discussions?	
	What does it take for you to learn the most?	Indiv./cooper. Silence/sounds Home/school Help from an adult…?	
How does the student deal with challenges?	What do you do if there is a task you cannot solve?	For how long do you attempt to solve it yourself?	
To what degree does the student seek challenges?	If you could choose between an easy task, a difficult task or one in between, which one would you rather do?	Do you like a challenge or would you rather finish quickly?	
What are the student's beliefs about the nature of a mathematical task, specifically concerning time? Does the student see perseverance as part of mathematical activity?	How long does it usually take to solve a task in math? How long do you think it is reasonable to work on a solution for a task?		
Beliefs about self			
How does the student perceive his/her own performance level + criteria for success	Do you think that you are good at math?	How can you fell that you are good/not good?	
Positioning	Do you sometimes help the other students?	Is it certain students, you help? Do they ask for your help?	
Independence	Do you sometimes get help from other students?	Do you ask for that help yourself?	
How does the student perceive him/herself in cooperative situations?	What do you think it is like to cooperate with you?		
What are the student's plans for the future	What do you think you can use mathematics for?	Both now and in the future	

	Do you think you will be dealing with math in the future?	In which way?	
	Do you have an idea which job or education you would like to have?	Is there anything you dream of?	
Beliefs about the application of mathematics (OJ1)			
What are the student's beliefs about the importance of mathematics?	All over the world, mathematics is taught in school. How do you think that is?	If education/job: Clarify where the mathematics is.	
Which experiences does the students have with the application of mathematics?	Do you know anyone who uses math?	Who? How do they use it?	
What beliefs does the student have about the application of mathematics in everyday life?	Can you think of anything that you or your parents use math for?	Other than shopping…	
What beliefs does the student have about the application of mathematics in society?	Can you think of anywhere else where mathematics is used?	In society/Denmark/th e world?	
	In the questionnaire, you were asked if mathematics is more important now than 100 years ago – can you elaborate on what you think about that?		
Beliefs about the historical development of mathematics (OJ2)			
What are the student's beliefs about origin of mathematics?	The mathematics that is in the textbooks – where do you think that comes from?	Has someone invented it? Or did someone just discover it? E.g., the coordinate system – where do you think that comes from?	
What are the student's beliefs about the time horizon of mathematics?	When do you think mathematics was used for the first time?	Why do you think so?	
Beliefs about the nature of mathematics as a subject area (OJ3)			
What are the student's beliefs about the	Can you say a little bit more about what the	Are there different ways of working?	

difference between mathematics and other subjects?	difference is between mathematics and other school subjects?	Are the tasks different? Do you speak differently in mathematics? Do you have to be good at other things in mathematics than in other subjects?	
Which criteria for success does the student assign to mathematics?	When are you good at math?	Or: what does it take to be good at math? Follow-up on checked boxes in questionnaire (quest. C2)	
Which working methods does the student believe to lead to success in mathematics?	*How* do you become good at math?	Can everybody become good at math?	
What are the student's beliefs about mathematics as a job?	Do you know what a mathematician is? What do you think he or she does?		
Mathematical activity / Drawing			
What are the student's beliefs about mathematical activity and/or oneself as a learner of mathematics?	*Ask about details in drawing*		

Outro: I don't have any more questions. It was really interesting to hear, what you had to say.
Is there anything else, you would like to tell me?
Thank you very much for your help.

Interview guide, students 2021

(Questions that were not part of the 2019 interview guide is in red)

Name: Class: Date:

Research questions	Interview questions	Follow-up	Individual notes
General attitude towards math			
What are the student's feelings and attitudes towards mathematics?	How do you feel about math at the moment?	What do you like the best to do in math?	
How does the student perceive the intervention?	Have you become more aware of your class' participation in this project?	What are your thoughts about the purpose of the project?	
Learning behavior			
What is the student's perception of own learning behavior?	How do you work in math at the moment? Do you feel that you are active in class? How did you work during the periods of lockdown?	How do you work individually? What do you think it is like to work with you in a group? Do you participate in class discussions? How has it been to return to school?	
To what degree does the student seek challenges?	If you could choose between an easy task, a difficult task or one in between, which one would you rather do?	Do you like a challenge or would you rather finish quickly?	
How does the student deal with challenges?	What do you do if there is a task that you cannot solve?	How long do you attempt to solve it yourself?	
What are the student's beliefs about the nature of a mathematical task, specifically concerning time? Does the student see perseverance as part of mathematical activity?	How long does it usually take to solve a task in math?	How long do you think it is reasonable to work on a solution for a task?	

Beliefs of self and social context			
How does the student perceive his/her own performance level + criteria for success	Do you think that you are good at math?	How can you fell that you are good/not good?	
How does the students look upon errors?	How do you feel about speaking in class discussions?	Are you nervous of making mistakes?	
How do the students perceive the learning environment in the class and the teaching of mathematics?	How is your class in relation to possibility of learning?	What are your thoughts about the way math is taught in you class?	
How does the students perceive cooperative work?	What does it take for a cooperation to function?	What is it like to work with you in math?	
What does the student imagine using math for in the future?	What do you think that you can use math for?	Now and in the future	
	Do you think that you will be dealing with math when you grow up?	In which way?	
	Do you have an idea what job or education you would like to have?	Is there anything you dream of?	
Beliefs about the application of mathematics (OJ1)			
What are the student's beliefs about the importance of mathematics?	Is mathematics important?	For you? For others? For society? For the world? If education/job: Clarify where the math is.	
What beliefs does the student have about the application of mathematics in everyday life?	Can you think of anything that you or your parents use math for?	Other than shopping...	
What beliefs does the student have about the application of mathematics in society?	Can you think of anywhere else where mathematics is used?	In society/Denmark/the world?	
	In the questionnaire, you were asked if mathematics is more important now than 100 years ago – can you elaborate on what you think about that?		

Has the intervention been successful in regard to presenting examples of the application of mathematics?	Do you feel that you learn about the application of mathematics in school?	Do you know more about it now than in the beginning of 6th grade?	
Beliefs about the historical development of mathematics (OJ2)			
What are the student's beliefs about origin of mathematics?	The mathematics that is in the textbooks – where do you think that comes from?	Has someone invented it? Or did someone just discover it?	
What are the student's beliefs about the time horizon of mathematics?	When do you think mathematics was used for the first time?	Why do you think so?	
What are the student's beliefs about the historical development of mathematics?	Has math change since then?	How might that be?	
Beliefs about the nature of mathematics as a subject area (OJ3)			
What are the student's beliefs about the characteristics of mathematics?	What is the subject of mathematics about, according to you?	What does math have in common with other subjects? How is math different from other subjects? Are there for example different ways of working?	
Which criteria for success does the student assign to mathematics?	When are you good at math?	Or: what does it take to be good at math? Follow-up on checked boxes in questionnaire (quest. C2)	
Which working methods does the student believe to lead to success in mathematics?	*How* do you become good at math?	Can everybody become good at math?	
What are the student's beliefs about mathematics as a job?	We have previously talked about what a mathematician does. Have you become wiser about that?		
The project			
Which impact has the intervention made on the student and the student's learning?	What do you remember best from the last two years of mathematics lessons?		

	What do you think you have benefitted the most from? What has been fun? What has been educational? What has worked well in relation to learning (methods, organization, etc.)?		