TOWARD MARVELS IN DYNAMIC GEOMETRY TEACHING AND LEARNING

Developing guidelines for the design of didactic sequences that exploit potentials of dynamic geometry to foster students' development of mathematical reasoning competency

Danish title

Mod mirakler i dynamisk geometri undervisning
Udvikling af guidelines for design af undervisningsforløb, der udnytter potentialer med dynamisk geometri til at fremme elevers udvikling af matematisk ræsonnementskompetence

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III. Højsted, I. H., & Mariotti, M. A. (2020a). *Analysing signs emerging from students' work on a designed dependency task in dynamic geometry*. Unpublished manuscript.

IV. Højsted, I. H. (2020c). A "toolbox puzzle" approach to bridge the gap between conjectures and proof in dynamic geometry. In A. Donevska-Todorova, E. Faggiano, J. Trgalova, Z. Lavicza, R. Weinhandl, A. Clark-Wilson, & H. G. Weigand (Eds.), *Proceedings of the 10th ERME Topic Conference (ETC10) Mathematics Education in the Digital Age (MEDA), September 16–18, 2020, Linz, Austria* (pp. 215–222). Johannes Kepler University. https://hal.archives-ouvertes.fr/hal-02932218

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III. English summary

Digital technologies are gaining an increasingly prominent role in Danish primary and lower secondary school mathematics education. The expectations are that digital technologies can support students' development of mathematical competencies. However, studies have demonstrated that the insertion of digital technologies into school mathematics does not automatically correspond to increased student learning outcomes (OECD, 2015). In fact, digital technologies can both produce marvels and cause disasters in mathematics teaching and learning (Niss, 2016). How digital tools are implemented seems to be essential. It is, therefore, a crucial research objective to unveil how didactic sequences may be designed for the efficient implementation of digital tools.

The overarching purpose of this dissertation is to identify guidelines for the design of didactic sequences that exploit potentials of dynamic geometry environments (DGEs) in relation to supporting students' development of mathematical reasoning competency in Danish lower secondary school.

The dissertation consists of five papers and this summarizing report – which, altogether, seeks to shed light on the abovementioned purpose by first examining what the potentials of DGEs in relation to reasoning competency are according to previous mathematics education research (paper I) and then examining the extent to which these potentials are currently utilized in Danish lower secondary school (paper II). Based on these initial studies, guidelines are identified for the design of didactic sequences that utilize these potentials to support students' development of mathematical reasoning competency (papers I, III, IV and V).

Methodologically, a mixed methods approach was applied with a qualitative priority. The quantitative data were collected from a web-based questionnaire that was developed and distributed to lower secondary school teachers. Anchored in design-based research methodology, the qualitative data were collected in connection with the design, testing and redesign of a didactic sequence in five different school classes. The data that were collected in the iterations of design are screencast recordings, videos, interviews, audio recordings and students' written products.

The results from the project can coarsely be summarized as follows: In the review of mathematics education research, four DGE potentials were identified in relation to mathematical reasoning competency: dragging, feedback, measuring and tracing (paper I). The results of the survey indicate that these potentials are only exploited to a limited extent in lower secondary schools (paper II). The utilization of the potentials is described in guidelines that comprise a learning trajectory in terms of

students' cognition, task design and the role of the teacher. The a priori guidelines that are theoretically developed initially (paper I) are subsequently refined empirically based on the data that emerged in the design-based research (papers III, IV and V), leading to a posteriori guidelines (Chapter 5).

The dissertation was completed in the period 1 January 2018–31 December 2020.

IV. Dansk resume

Digitale teknologier får en stadig mere fremtrædende rolle i grundskolens matematikundervisning. Forventningerne er at digitale teknologier kan understøtte elevers udvikling af matematiske kompetencer. Imidlertid viser studier at brugen af digitale teknologier ikke garanterer større læringsudbytte (OECD, 2015). Faktisk kan digitale teknologier både bidrage til mirakler, men også forårsage katastrofer i matematikundervisningen, og det lader til at være af essentiel betydning hvordan undervisningen omkring IT-værktøjet designes (Niss, 2016).

Formålet med denne afhandling er at identificere principper for design af undervisning, der udnytter potentialer ved dynamiske geometri miljøer i forhold til at understøtte elevers udvikling af matematisk ræsonnementskompetence i grundskolens udskoling.

Afhandlingen består af fem papers og denne kappe, som tilsammen søger at belyse det ovennævnte overordnede formål ved først at undersøge hvilke potentialer der er ved DGE ifølge tidligere matematikdidaktisk forskning (paper I), for derefter at undersøge i hvilken udstrækning disse potentialer udnyttes aktuelt i grundskolens udskoling (paper II). Med udgangspunkt i disse indledende studier identificeres guidelines til design af undervisningsforløb der udnytter disse potentialer til at understøtte elevers udvikling af matematisk ræsonnementskompetence (paper I, III, IV, V).

Metodisk anlægges et mixed methods design med kvalitativ prioritet. De kvantitative data indsamles i form af et web-baseret spørgeskema som blev udviklet og distribueret til udskolingslærere. Forankret i design-based research metodologi indsamles de kvalitative data i forbindelse med design, test og re-design af et undervisningsforløb i fem forskellige skoleklasser. De data, der indsamles i designiterationerne, er screencast-optagelser, video, interviews, lydoptagelser og de studerendes skriftlige produkter.

Resultaterne fra projektet kan groft sammenfattes som følger:

I gennemgangen af matematikdidaktisk forskning, identificeres fire DGE-potentialer i relation til matematisk ræsonnementskompetence; dragging, feedback, måling og sporing (papir I). Resultaterne fra spørgeskemaundersøgelsen viser, at disse potentialer kun udnyttes i begrænset omfang i grundskolens udskoling (papir II). Udnyttelsen af potentialerne er beskrevet i guidelines, der omfatter en læringsbane med hensyn til elev kognition; opgave design og lærerens rolle. A priori guidelines, der oprindeligt er teoretisk udviklet (paper I), videreudvikles efterfølgende empirisk, baseret på de data, der fremkommer i design-based research delen (paper III, IV og V), hvilket fører til a posteriori guidelines (kapitel 5).

Afhandlingen er gennemført i perioden d. 1 januar 2018 til d. 31 december 2020.

1. Introduction

1.1. Problem and motivation

Over the last decade, dynamic geometry environments (DGEs hereinafter) – in particular, GeoGebra (https://www.geogebra.org/) – have become popular and extensively used in Danish primary and lower secondary school (Blomhøj, 2016; Jankvist et al., 2018; Ministeriet for børn, undervisning og ligestilling, 2016). In fact, in 2018, Denmark was the country in the world with the most unique users on the GeoGebra website relative to the size of the population (M. Hohenwarter, personal communication, April 4, 2018).

Studies have shown that DGEs contain potentials that may be exploited in activities to support students' development of mathematical reasoning (e.g., Arzarello et al., 2002; Baccaglini & Mariotti, 2010; Edwards et al., 2014; Laborde, 2001; Leung, 2015). From this perspective, the trend of increased DGE usage in Denmark may be considered a positive development, especially in light of the fact that students' reasoning abilities are apparently underdeveloped in Denmark (Jessen et al., 2015; Larsen & Lindhart, 2019). However, even if DGE usage is widespread, students' accessibility to digital tools, such as DGEs, does not guarantee a greater learning outcome (OECD, 2015). In fact,

> the very same piece of digital technology can give rise to "marvels" as well as to "disasters" in mathematics education. This means that no ICT system, hard or soft, is, in and of itself, good or bad for mathematics education. (Niss, 2016, p. 247)

The way that DGE is used, and the way students appropriate it, seems to be essential (Jones, 2005).

The *motivation* underlying this project partly stems from the hypothesized *problem* that the utilization of DGE potentials in relation to mathematical reasoning is limited in Danish primary and lower secondary school, despite the fact that DGEs (GeoGebra in particular) are quite popular.

The Danish KOM framework[1] (Niss & Højgaard, 2011, 2019; Niss & Jensen, 2002),[2] which is used in the Danish curriculum, describes the development of mathematical reasoning in terms of possession of mathematical reasoning competency. The notion of mathematical competencies has gained growing attention in mathematics education research all around the world, although

[1] The KOM framework, an acronym for "competencies and mathematical learning", was published in Danish in 2002 and in English in 2011.

[2] Note that Jensen and Højgaard are the same author who changed their surname between publications.

conceptualizations of the notion differ (Niss et al., 2016). According to Niss et al. (2016), it is important that we understand and develop educational practices that may support students' development of mathematical competencies.

> Fostering, developing and furthering mathematical competencies with students by way of teaching is a crucial […] priority for the teaching and learning of mathematics [...] We now need to understand the specific nature of the contexts and other factors that help create such progress. (Niss et al., 2016, p. 630)

Departing from this point of view, it is important that we understand the specific factors that are involved in the context of DGE teaching/learning when the educational aim is to support students' development of mathematical reasoning competency. It is important to unveil how thoughtfully designed DGE teaching/learning sequences can contribute to creating "marvels" in Danish lower secondary school mathematics, as well as around the world. That is the endeavor that this project aims to contribute toward.

1.2. A brief historical view on DGE internationally and in Denmark

Since the introduction of DGEs in the early 1980s, the programs have assumed an increasing presence in mathematics classrooms in countries all over the world and now play a nontrivial role in mathematics teaching and learning at several educational levels. The very first program, the Geometric Supposer, was developed by Judah L. Schwarz and Michal Yerushalmy in the early 1980s at MIT's Educational Development Center, and it could run on Apple II (Oldknow, 1997). Early educational research studies on the Geometric Supposer highlighted the possibility for students to create geometry that could complement the traditional emphasis in geometry teaching on the comprehension of deductive mathematical systems and the fact that it allowed students to efficiently test conjectures with great speed (Yerushalmy & Houde, 1986). The Geometric Supposer allowed students to study predefined shapes but did not have draggable objects – which was the next step in the evolution of programs that came with Cabri Géomètre, developed by Jean-Marie Laborde in the mid-to-late 1980s (http://www.cabri.net/cabri2/historique-e.php; Laborde & Laborde, 1995), and Geometer's Sketchpad, created by Nicholas Jackiw (Jackiw, 1991). Cabri Géomètre was widespread in France and the UK, while Geometer's Sketchpad was prevalent in the US (Oldknow, 2001). These programs, which were similar in many aspects, provided construction tools of objects such as points and lines, as well as tools that could be used in the Euclidian plane, such as an angle bisector or reflections in a line, and granted the possibility of measuring figures and making calculations with them. One difference between the programs was that in Cabri, the user would select the operation first and then the objects to which the operation should be performed, whereas in Geometer's Sketchpad, you would first select the objects and then the operation (Oldknow, 2001). Several other programs were developed in the years to come – for example, Geometry Inventor (Clements, 1995), Thales (Fabricz et al., 1990) and Cinderella (http://www.cinderella.de; Richter-Gebert & Kortenkamp, 2000). As the hardware platforms improved quickly over time, so did the affordances of the DGE software. GeoGebra was developed by Markus Hohenwarter in 2001/2002 during his master's thesis (Hohenwarter, 2002), and it offered a new bidirectional combination of DGEs and computer algebra systems (CAS) allowing for a closer linkage of visualization potentials, which were separated programs in previous software (Hohenwarter et al., 2009; Hohenwarter & Fuchs, 2004). According to Hollebrands and Lee (2012), more than 40 DGEs had been developed by 2012.

In Denmark, Viggo Sadolin developed the program GeomeTricks (Sadolin, 1997), which was sold to more than 600 schools and translated to 10 languages, however, development on the program stopped. A study from 2009 showed that 90% of the teachers in primary and lower secondary schools in Denmark rarely or never used DGEs in their teaching (Vejbæk, 2011). Nonetheless, as mentioned previously, DGE usage has grown rapidly since then, and there are signs that DGEs have become a visible part of mathematics education in Danish primary and lower secondary school (Blomhøj, 2016; Jankvist et al., 2018; Ministeriet for børn, undervisning og ligestilling, 2016). This may in part be due to the fact that substantial economic resources have been used in Denmark over the last decade to boost the usage and accessibility of digital tools in education (Undervisningsministeriet, 2015).

Another plausible explanation for the increased usage of DGE can be found in the establishment of the Danish GeoGebra Institute in 2009, whose aim was to promote digital tools integration in mathematics teaching by developing free GeoGebra resources and offering courses for teachers, as well as to build a national network of experts to further support GeoGebra integration (Misfeldt & Andresen, 2010). The institute was connected to the International GeoGebra Institute (Hohenwarter & Lavicza, 2007), which aimed to establish national self-sustaining local user groups (Hohenwarter & Lavicza, 2011).

The increased usage of DGEs may also be attributed to the curricular development in Denmark. In the curriculum *Klare Mål*, "Clear Objectives" (Uddannelsesstyrelsen, 2001), which included binding final objectives for the central areas of knowledge and skills that were demanded at the end of the teaching of mathematics in primary and lower secondary schools, dynamic geometry was not mentioned at all. The curriculum was revised in 2009 with the *Fælles Mål 2009 Matematik* "Common Goals 2009 Mathematics" (Undervisningsministeriet, 2009). In the new curriculum, the use of DGEs is not mentioned as a part of the binding final objectives of required knowledge and skills but is mentioned 22 times as a suggestion for certain, primarily geometrical, activities. The curriculum was revised again in 2014 (Undervisningsministeriet, 2014), and the use of DGEs was mentioned more directly in the learning goals as a guiding learning aim so that students, already after having completed grade 3, should have "knowledge of methods for drawing simple planar figures, including with a dynamic geometry program" (Undervisningsministeriet, 2014, p. 9). Digital tools and DGEs are similarly mentioned in the learning goals for grade 6 and grade 9 (Undervisningsministeriet, 2014).

1.3. Previous research on DGE and reasoning competency

An elaborated account of previous research on dynamic geometry environments in relation to mathematical reasoning competency is presented in the first paper (Højsted, 2020a), to which the reader is referred, however, a few words are delivered here to set the scene.

While several researchers have studied the effects of introducing DGEs in mathematics education in terms of students' cognition (e.g., Arzarello et al., 2002; Jones, 2000; Laborde, 2005b), fewer have focused on the teaching involved in successfully implementing DGE activities for certain educational goals. According to Komatsu and Jones (2018), the role of the teacher in this implementation process is, in fact, an understudied topic, and the same applies to the study of DGE task design related to specific learning aims (see also Sinclair et al., 2016). Yet advances have been made – with some researchers outlining task design models or principles developed in relation to DGE usage for particular learning aims, such as exploration, reasoning, conjecturing and proving (see Baccaglini-Frank et al., 2017; Fahlgren & Brunström, 2014; Leung, 2011; Lin et al., 2012; Komatsu & Jones, 2018; Olsson, 2019; Sinclair, 2003). Others have developed models in order to evaluate the quality and suitability of tasks in relation to the affordances provided by DGEs (Trocki, 2014; Trocki & Hollebrands, 2018). While the role of the teacher is understudied, a pertinent contribution in this area is that of the Theory of Semiotic Mediation (TSM) (Bartolini-Bussi & Mariotti, 2008), which highlights the role of the teacher in supporting students' development of mathematical meanings as they are working in activities that are centered around the use of artefacts, such as a DGE. This project is theoretically anchored in the TSM – which is, therefore, described in more detail in Chapter 2.

Even though several studies deal with issues that concern DGEs and, in some cases, learning aims related to reasoning competency in one way or another, there are no previous studies that have specifically aimed at supporting the implementation process concerning the utilization of DGE potentials to support students' development of mathematical reasoning competency.

1.4. The issues addressed by the dissertation – research questions

Acknowledging the issue raised by Niss et al. (2016) on the need for research to understand the specific factors and contexts that can foster the development of mathematical competencies and to address the hypothesized problem of the low utilization of DGE potentials in relation to reasoning competency, this project seeks to understand how teaching with DGEs can be designed to utilize its potentials in relation to fostering students' development of mathematical reasoning competency in the context of Danish lower secondary school teaching.

This endeavor is guided by three overarching research questions, of which the first and second may be considered as auxiliary questions that are posed in the process of answering the third and main research question.

Since the aim of the project revolves around the utilization of DGE potentials in relation to reasoning competency, it is pertinent to investigate what these potentials comprise. The meaning of "potential" in this project is affordances of DGEs – which are not available in other typical mathematics education tools, such as paper and pencil. In that light, the first research question is:

1. *What are the potentials of DGEs in relation to supporting students' development of reasoning competency?*

As mentioned in section 1.2, the motivation to conduct this research study partly stems from an underlying hypothesis that the potentials of DGEs, such as GeoGebra, are not utilized in Danish lower secondary schools, even if DGE software is indeed popular. The hypothesis is that DGEs are predominantly used as a substitution for the paper and pencil environment. To investigate this hypothesis, the second research question is put forward:

2. To what extent are the potentials currently utilized in Danish lower secondary school?

Answering the second question is not only interesting in relation to the mentioned hypothesis, it may also give insights that are valuable in the design of teaching, which is on the agenda in the third research question. Based on the work produced in questions one and two, the main research question of the project may be formulated.

3. ***Which research-based guidelines feature in the design of teaching that utilizes DGE potentials in order to support students' development of reasoning competency?***

1.5. The structure of the thesis – chapters and papers

In addition to the papers, the thesis consists of seven chapters that outline the research project, acting as a summarizing report around the papers. The main objective of the chapters is to disseminate a coherent report of the whole research project, which the papers by themselves do not, and to elaborate on issues that are not described in detail in the papers, such as a thorough unfolding of the methodological and theoretical considerations and discussions of these, as well as the results. Additionally, the summarizing report aims to elaborate on the relationship between the individual papers and their placement in the project. A brief overview of Chapters 2–7 is presented next, followed by an overview of the papers in the thesis.

Chapter 2 is devoted to elaborating upon two of the major conceptual frameworks that the project draws upon: the KOM framework and its reasoning competency (Niss & Højgaard, 2011) as well as the TSM (Bartolini-Bussi & Mariotti, 2008). These frameworks are described to some extent in the papers; however, the journal-based paper format presents limitations for in-depth descriptions of the kind that is put forward in this chapter. After the elaboration, the research questions are revisited in light of the conceptual frameworks. At the end of Chapter 2, other theoretical constructs that are utilized in the project are briefly mentioned.

Chapter 3 portrays the three types of methodological approaches applied in the project, the hermeneutic approach to literature reviewing, the quantitative approach deployed for the survey and the design-based research (DBR) approach. The chapter outlines how these methodological choices can address the three research questions and provides an account of the considerations and choices made and how that is reflected in each paper. The chapter also describes the data collection process, as well as how the data were analyzed. Finally, ethical considerations are elaborated upon.

Chapter 4 describes the results of the research, referring to the results from each of the papers. The chapter concludes with a section that summarizes the results.

Chapter 5 unfolds a discussion that involves discussing the results and proposing refinements to the a priori guidelines based on the empirical outcome, as well as producing a synthesis in the form of a condensed model. The suitability of the methodological and theoretical choices is reflected upon in light of the results that they produced. The relationship between the individual papers, and their relationship to the overall project, is also considered. Validity and reliability are also discussed using notions from DBR, as are the notions of generalizability and trustworthiness.

Chapter 6 delivers a concise conclusion to the project by reiterating the research questions and referring to the results that answer the questions.

Chapter 7 shares some considerations concerning the novelty of the research results obtained, referring to other research in the field, and finally, some comments concerning dissemination to praxis are shared.

Following the chapters, the thesis contains five research papers, all of which contribute to dealing with the research questions. A short description of the papers is provided below, describing the aim of the papers and their contribution to the overall research project.

Paper I is entitled *Guidelines for Utilizing Affordances of Dynamic Geometry Environments to Support Development of Reasoning Competency*. It is based on a literature review that was conducted to unveil the potentials of DGEs in relation to reasoning competency, thereby addressing the first research question. As the title suggests, the focus of the paper is to not only present literature review results but also develop a priori guidelines based on them. The paper, therefore, partly contributes to answering the third research question. The paper is published in *Nordic Studies in Mathematics Education*. Notice that the table format guidelines are attached as an appendix in the paper due to page limits of the journal.

Paper II is entitled *Teachers Reporting on Dynamic Geometry Utilization Related to Reasoning Competency in Danish Lower Secondary School*. This paper elaborates on the development and analysis of a survey aiming to investigate to what extent the potentials, which were uncovered in the review, are utilized in lower secondary schools. The paper is directly relevant in answering the second research question; however, it also gives insights into Danish DGE teaching practice, which is used later in the project in answering the third research question. A short version of the paper was presented at the 14th International Conference on Technology in Mathematics Teaching in Essen, Germany, and published in the conference proceedings. It was developed into a journal paper afterward and published in *Digital Experiences in Mathematics Education*.

Paper III is entitled *Analyzing Signs Emerging from Students' Work on a Designed Dependency Task in Dynamic Geometry*, and it was coauthored with Professor Maria Alessandra Mariotti. The paper reports on the design principles used in the design of the initial tasks of a didactic sequence and, based on the data, evaluates the design. A brief earlier version of the paper was presented at Madif–12: the

12th research seminar of the Swedish Society for Research in Mathematics Education in Växjö, Sweden, and published in the conference proceedings. The paper was then further developed into a journal paper and submitted to the *International Journal of Mathematical Education in Science and Technology*, of which it is now in review.

Paper IV is entitled *A "Toolbox Puzzle" Approach to Bridge the Gap between Conjectures and Proof in Dynamic Geometry*. The paper sheds light on a type of task design that is intended to support students' development of inferential arguments. The paper was presented at MEDA2, the 10th ERME Topic Conference, Mathematics Education in the Digital Age, which was hosted online from Linz, Austria. The paper was published in the conference proceedings.

Paper V is entitled *Guidelines for the Teacher – Are They Possible?* and the present author coauthored it with Professor Maria Alessandra Mariotti. The paper reports on the design and implementation of teacher guidelines in the final iteration of the DBR study. The paper has been submitted to NORMA 20 – the Ninth Nordic Conference on Mathematics Education, which was postponed and is due to take place in Oslo, Norway, in June 2021.

2. Choosing conceptual frameworks

The choice of defining the mathematical aims of the project in the terminology provided by the KOM framework's description of competencies was made at the start of the project and is justified by two central arguments. First, the notion of competencies, or different conceptualizations of the idea of competencies, has gained increasing attention around the world due to considerations of the fact that it is not sufficient to define mathematics and mathematical expertise by means of mathematical subject matter only (Niss et al., 2016). Acknowledging the importance of aligning mathematics education with the teaching and learning of competencies, it, therefore, becomes an important research aim to understand the particular potentialities in different contexts in which competencies can be fostered, such as the context of DGE usage (Niss et al., 2016). Second, in Denmark, the KOM framework is integrated into the curriculum at many educational levels – including lower secondary schools, where the framework significantly influences the description of students' mathematical learning goals. For that reason, considering that this project is situated in the context of Danish lower secondary schools and concerns mathematical reasoning, it is deemed relevant and appropriate to define the learning aims in terms of the KOM framework and its reasoning competency.

The TSM (Bartolini-Bussi & Mariotti, 2008), was chosen – as it addresses issues related to the teaching and learning of mathematics in the context of digital tools, which are not expanded in detail in the KOM framework. The theory provides a framework for describing the complex relationship between artefacts; students' cognitive development; and the teacher's role. More specifically, the relationship between tasks performed with artefacts, such as DGEs, and students' development of mathematical meanings in relation to the artefact activity and the role of the teacher in supporting this development. During the review phase and development of paper I, the framework was found to be useful to theoretically anchor the project since this is exactly the focus of the project – with the artefact being a DGE that is used in activities to foster the development of mathematical meanings that are coherent with the constituents of reasoning competency – and, importantly, because the project also aims to focus on the role of the teacher. The choice of using the TSM is further elaborated in section 2.1.1., after the presentation of notions from the KOM framework.

2.1. The KOM framework and its reasoning competency

What does it mean to master mathematics? What are the constituents of mathematics as a subject? These are some of the questions posed and discussed by Niss (1999, 2000, 2002) around the turn of the millennia – which, ultimately, lead to the development of the Danish KOM framework (Niss & Højgaard, 2011, 2019; Niss & Jensen, 2002). The framework was a reaction, or one could say an attempt at a remedy, to a disease in mathematics education known as "syllabusitis" (Blomhøj & Jensen, 2007; Lewis, 1972). This notion of "syllabusitis" was discussed by Jensen (1995) as a disease that leads one to confuse the syllabus with actual expertise or competence. It is, according to Niss (1999, 2000, 2002), not sufficient to define mathematics and mathematical mastery by means of stating mathematical subject matter – that is, reducing mathematics to a list of topics in a syllabus.

> Niss: "It [subject matter] does not hit the nail on the head. It is, of course, a nail, but it does not hit the head of the nail. The crucial thing is something else. The crucial thing is the way you go about mathematics, and the approaches you have, and the ideas you have, and the abilities you have to behave in the mathematical situations, to act, that is. The KOM project is, after all, an action-oriented enterprise." (Sloth & Højsted, 2016, p. 80, translated from Danish)

To fight "syllabusitis" in mathematics education, the KOM framework attempts to describe what mathematical mastery entails across mathematical topics and educational levels, not in terms of mathematical subject matter (although this is, of course, an important constituent) but rather in terms of mathematical ability. It introduces a competency-based approach comprising eight mathematical competencies. The framework highlights what it means to be able to do mathematics and defines mathematical competency as "a well-informed readiness to act appropriately in situations involving a certain type of mathematical challenge" (Niss & Højgaard, 2011, p. 49). Although each of the competencies has its own identity, they are all interwoven – as illustrated in the so-called KOM flower (Figure 1), which elucidates that the competencies are not separate or clearly delineated but rather that there are overlaps between them. For example, reasoning competency is closely related to problem handling and modelling competencies, as it involves justification of decisions/actions within the mobilization of these competencies. The KOM flower is a visual representation of the competencies divided into two groups. Being able *to ask and answer in, with and about mathematics* characterizes the four competencies on the left side, while being able *to deal with the language and tools of mathematics* relates to the four competencies on the right side. In the KOM framework, the eight mathematical competencies are described separately and in relation to the mathematics teacher's work (Niss & Højgaard, 2011, pp. 83–109) – where examples, as well as didactic and pedagogical comments, are given. In general, the competencies have an active component and a passive component, where the passive side of a competence typically consists of being able to understand and follow when others use the competence, while the active part of mastering the competence involves being able to act according to its characteristics.

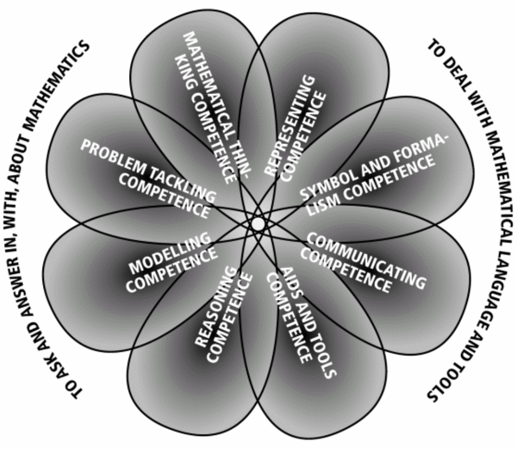

Figure 1: KOM flower (Niss & Højgaard, 2011, p. 1)

As mentioned previously, the competency that is in focus in this project is the **reasoning competency**. In the description of reasoning competency, **reasoning** itself is defined as "a chain of argument […] in writing or orally, in support of a claim" (Niss & Højgaard, 2011, p. 60). The reasoning competency is situated on the left side of KOM flower and therefore concerns the ability to ask and answer in, with and about mathematics. The active aspects of the reasoning competency comprise the ability to create and present **formal**, as well as **informal**, arguments, while the passive aspect refers to the ability to follow and evaluate **arguments made by others**. The competency involves

> understanding what a mathematical proof is and how this differs from other forms of mathematical reasoning, e.g. heuristics based on intuition […] understanding of the logic behind a counter example […] to uncover the basic ideas in a mathematical proof, including distinguishing between main lines of argument and details […] the ability to devise and carry out informal and formal arguments (on the basis of intuition) and hereby transform heuristic reasoning to actual (valid) proofs. (Niss & Højgaard, 2011, p. 60)

Niss and Højgaard (2011) further state that mathematical argumentation and mathematical reasoning, in general, are oftentimes juxtaposed with justification of mathematical theorems and that reasoning competency not only includes these aspects but also concerns the ability to assess "the validity of mathematical claims" in general, such as "the justification of answers and solutions," and being able to convince "yourself and others of the possible validity of such" (Niss & Højgaard, 2011, p. 60).

It follows from the description of reasoning competency that proof is a part of mathematical reasoning – that is, it is considered as a particular form of mathematical reasoning. However, there are different interpretations of the meaning of the notion of proof, which the KOM framework does not directly address. Mariotti (2012) puts forward two extremes in such interpretations: 1) proof as the product of theoretical validation of already stated theorems and 2) proof as the product of a proving process, which includes exploring and conjecturing, as well as proving conjectures. In the context of school mathematics, the notion of proof has largely shifted to the process of proving, partly because of the facilitation of experimentation by digital technologies (Sinclair & Robutti, 2013). NCTM (2008) offers a *reasoning and proof cycle*, which consists of the *exploration* of a mathematical problem or context, making a *conjecture* about the problem/context and, finally, providing *justification* for the conjecture. Even though the KOM framework does not address proof in this sense, the interpretation of proof as a process resonates with the emphasis in the KOM framework on the ability to act when faced with mathematical challenges – the ability to investigate and do mathematics. Therefore, in this thesis, proof is understood as a process, and DGE potentials in the proving process are included.

The degree to which a person (e.g., a student) possesses a competency is described according to three measures in the KOM framework (Niss & Højgaard, 2011). One measure is the *degree of coverage*, which indicates to what extent the student can mobilize the different aspects of the competency (e.g., both the passive aspect and the active aspect of the competency). Another measure concerns a student's *radius of action* in relation to the competency. It specifies the range of situations and contexts in which the student may be able to activate the competency. The third measure pertains to a student's *technical level*, which concerns how difficult the problem or situation at hand is and, therefore, how advanced the necessary activation of the competency is. If we exemplify these three measures in relation to reasoning competency, then we might say that a student who is able to follow an inferential argument put forward by his peer and yet is unable to produce and communicate such an argument by himself not only possesses some *degree of coverage* of reasoning competency but also lacks some *degree of coverage* since he currently only manages the passive aspect. If the student can string together coherent deductive steps to validate a conjecture in the area of geometry but cannot do so in the area of statistics, then we might say the student has a limited *radius of action*. The student could be able to follow and produce complicated and technically advanced reasoning, for example, in a complex proof with many steps utilizing several theorems and therefore can be considered to possess a high *technical level*. It is clear that the measures have a subjective character because ascertaining what constitutes a high technical level is, of course, relative to the precondition of the students, and it depends, in particular, on the student's age and peers. What is considered as a high technical level in grade 1 may be considered as trivial in grade 9 and similarly between students of the same age in different countries.

The KOM framework is integrated into the curriculum design at most levels of mathematics education in Denmark. The current curriculum for primary and lower secondary school "Fælles Mål" – therefore describes the required learning goals in terms of expected possession of mathematical competencies, as well as the subject matter that the competencies should be mobilized in relation to (Undervisningsministeriet, 2014). The competency approach is, however, not only limited to Denmark, as it has previously shaped the foundation for the PISA assessment and analytical framework for mathematics (OECD, 2017) and has had an impact on mathematics education around the globe (for a detailed account, see Niss et al., 2016).

Niss and Højgaard (2019) published an updated version of the framework, which included some revisions to the description of reasoning competency. In the revised version, the authors highlight and elaborate upon mathematical argumentation, while proof plays a less prominent role. However, the constituents of reasoning competency remain the same (Niss & Højgaard, 2019). I will mainly refer to reasoning competency as it was described in Niss and Jensen (2011) in this thesis because much of the work in this Ph.D. project started before the revised version of the KOM framework was published.

2.1.1. KOM and digital tools

The KOM framework does not allocate much attention explicitly to digital tools in the teaching and learning of mathematics (Højsted et al., 2020). It does, however, include two more aspects that are relevant and worth mentioning in relation to the aims of this project. First, there is the *aids and tools competency*, whose characteristics are "knowing [the] possibilities and limitations of, and being able to use, aids and tools" (Niss & Højgaard, 2011, p. 68), referring not only to digital tools but also to all tools that are used in mathematics education. While the tools and aids competency does describe that students need to be able to use and know about strengths and weaknesses of digital tools, it offers limited analysis in relation to the acquirement of the competency (Jankvist et al., 2018). Second, the KOM framework also lists six mathematics teacher competencies, which outline "a range of specific mathematical, didactic and pedagogical competencies" (Niss & Jensen, 2011, p. 85). The teacher competencies outline the practice of a mathematics teacher in broad terms and do not directly address teaching that involves the use of digital tools (Højsted et al., 2020). Since the project goals relate to exactly that – teaching and learning with DGEs – it is necessary to complement the KOM perspective with theoretical constructs that can shed light on the specific context of mathematics teaching and learning with digital tools.

2.2. The TSM

In the TSM, Bartolini-Bussi and Mariotti (2008) expand on the relationship between the use of artefacts, such as DGEs, and students' cognitive development from a Vygotskyan point of view (Vygotsky, 1934/1978). Bartolini-Bussi and Mariotti (2008) discuss the notion of artefacts, referring to the distinction put forward in Rabardel's (1995) instrumental approach between an artefact and an instrument. An artefact is a material or symbolic object (e.g., a DGE) designed to be used for specific purposes. This implies that certain knowledge is necessary to use the artefact according to the purposes for which it was designed. The notion of an instrument includes exactly this cognitive part that enables the usage of the artefact for a specific purpose. The instrument comprises both artefact components and cognitive utilization schemes, which empowers the subject to use the artefact for some purpose; hence, it becomes an instrument for the subject for a certain class of situations (Vérillon & Rabardel, 1995). Rabardel (1995) denotes the evolution of artefacts into instruments, which can be a long and complex process, as instrumental genesis. The instrumental approach has proven powerful to outline important aspects related to students' behaviors in artefact-centered activities; however, Bartolini-Bussi and Mariotti (2008) point out,

> The instrumental approach has to be further elaborated in order to match the complexity of classroom activity and in particular that of the teaching learning of mathematics, in fact it may provide a frame to analyse the cognitive processes related to the use of a specific artefact and consequently what will be considered its semiotic potential. (p. 749)

Hence, complementing the cognitive perspective in the instrumental approach and matching the complexity of mathematics classroom activity, the TSM attempts to characterize how teachers can utilize the possible ways of using an artefact, such as a DGE, to support the teaching and learning process (Bartolini-Bussi & Mariotti, 2008).

TSM builds on Vygotsky's (1934/1978) notion of semiotic mediation. From this perspective, the construction and usage of artefacts is embedded in human activities, and these artefacts contribute not only to solving tasks but also at the level of cognition (i.e., shaping ways of human thinking). Vygotsky hypothesized that cognitive development is advanced through two "lines": the natural line, which pertains to elementary mental functions, and the social/cultural line, which comprises the higher mental functions. The social/cultural line is interesting from the point of view of mathematics learning in school, and particularly two notions coined by Vygotsky that are related to cognitive development are of interest: the zone of proximal development (ZPD) and internalization (Vygotsky, 1934/1978). According to the notion of the ZPD, development of higher mental functions is possible because of a collaboration between individuals occupying asymmetrical roles in a social setting in relation to knowledge – for example, the asymmetrical relationship between students and the teacher in relation to mathematical knowledge. Cognitive development of higher mental functions is described as a process of internalization. The internalization process has

> two main aspects: it is essentially social; it is directed by semiotic processes. In fact, as a consequence of its social nature, external process has a communication dimension involving [the] production and interpretation of signs. That means that the internalization process has its base in the use of signs. (Bartolini-Bussi & Mariotti, 2008, p. 750)

The basic didactic hypothesis of the TSM is that signs, which are produced from activities with an artefact, are socially elaborated. Signs are to be interpreted broadly here – referring to not only any type of signs that a subject might produce, most commonly natural language, but also gestures such as pointing, clicking on a computer screen (e.g. on icons in GeoGebra) or developing written signs (including sophisticated semiotic systems, such as the formalism commonly produced by mathematicians).

The TSM addresses students' initial production of artefact signs as the artefact is used and on the following transformation into mathematical signs. This distinction of these two types of signs is made to highlight that the **personal meanings** underlying the artefact signs initially produced by the student do not necessarily match the **mathematical meanings**, which an expert mathematician (the teacher) would recognize. However, through social interaction, the mathematics teacher can mediate the evolution from personal meanings to mathematical meanings and thereby support the students' production of mathematical signs.

Bartolini-Bussi and Mariotti (2008) discuss the fact that working with an artefact to solve a task can produce both personal meanings and mathematical meanings and use the notion of the semiotic potential of an artefact to describe the duality of meanings that may emerge. The semiotic potential of an artefact may be exploited in order to guide the evolution of mathematical meaning, consistent with the educational goal of a didactic sequence. Specifically, an artefact activity may be followed by a classroom discussion in which the teacher may identify the students' personal signs and interpret their underlying meaning and then by means of social interaction – for example, by posing questions and highlighting certain student answers, the teacher may mediate the development of mathematical meanings.

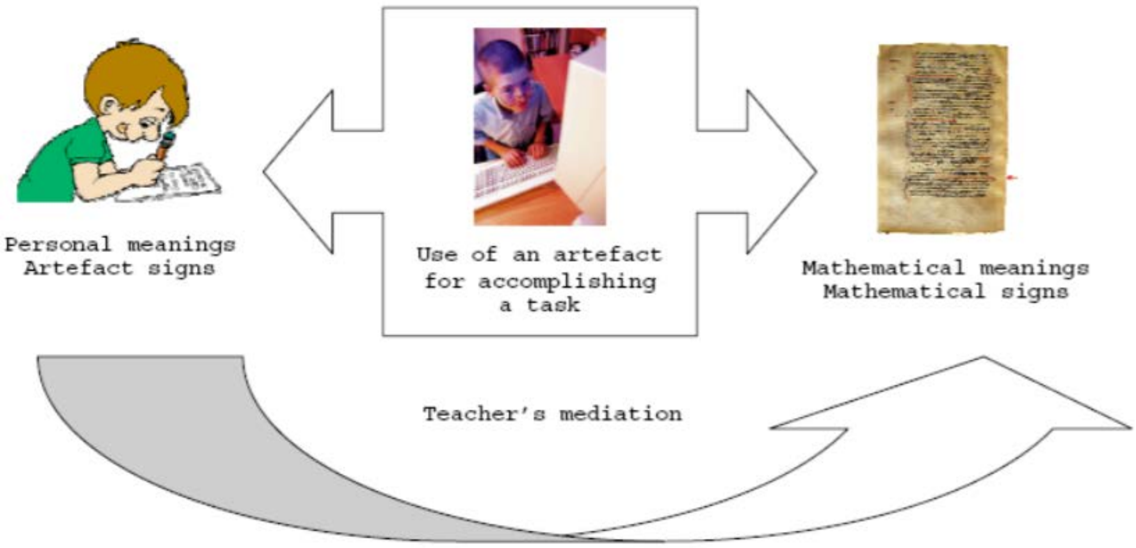

Figure 2: Semiotic potential of an artefact and the teacher's mediation (Mariotti & Maracci, 2011)

When a teacher is aware of the semiotic potential of an artefact (i.e., the duality of meanings it may provoke) and the artefact is used in a designed didactical sequence to exploit its semiotic potential, then the teacher uses the artefact as a tool of semiotic mediation (Bartolini-Bussi & Mariotti, 2008).

Mariotti (2012) emphasizes that the analysis of the semiotic potential of an artefact may be considered as the core of any teaching design. She describes how the exploitation of the semiotic potential includes the following:

> The orchestration of didactic situations, where students face designed tasks that are expected to mobilise specific schemes of utilisation, and consequently, situations in which students are expected to generate personal meanings. […] the orchestration of social interactions during collective activities, where the teacher has a key role in fostering the semiotic process required to help personal meanings, which have emerged during the artefact-centred activities, develop into the mathematical meanings that constitute the teaching objectives. (Mariotti, 2012, p. 170)

From this perspective, the aim of the first and third research questions of uncovering DGE potentials in relation to reasoning competency and developing guidelines to utilize these potentials can be related to analyzing the semiotic potential of DGEs in relation to reasoning competency and elaborating upon the possible exploitation of the uncovered semiotic potential.

2.2.1. The didactical cycle

Once the a priori analysis of the semiotic potential of an artefact in relation to a certain learning goal (e.g., reasoning competency) has been conducted, and a didactic sequence has been designed with the aim of exploiting this semiotic potential, then the TSM framework also models the teaching and learning process – that is, the evolution of personal meanings into mathematical meanings. According to the TSM, this evolution can occur through iterations of didactic cycles that each contain three different types of activities (Mariotti & Maracci, 2011). The first type is *activities with the artefact* that are the initial activities at the beginning of each cycle in which the students work with the artefact to solve tasks. The task-based activities are designed to intentionally foster the production of signs that are linked to the use of the artefact. This is followed by the second type of activity, *activities of individual writing*, in which the students are requested to engage in semiotic activities aiming to provide written productions that reflect on the artefact-centered activities that they previously engaged in. For example, the students can be requested to describe what they experienced when they solved a task that requested them to drag certain geometrical objects in the DGE. These written productions can then become the point of departure in the discussions that follow afterward in the classroom. The third type of activity concerns exactly that – *classroom discussions*, which are of utmost importance in the TSM because the development of mathematical reasoning competency is, as mentioned previously, considered as the development of higher mental functions (which, according to the ZPD, is possible in a collaboration between individuals occupying asymmetrical roles in relation to knowledge in social interactions). The main role of the teacher in this activity is that of fostering the students' evolution toward mathematical meanings, which might be done by analyzing and discussing the students' written texts and highlighting various solutions in collective discussions. The objective is to develop "shared meanings having an explicit formulation de-contextualized from the artefact use, recognizable and acceptable by the mathematicians' community" (Mariotti & Maracci, 2011, p. 94).

2.2.2. Teacher actions

Bartolini-Bussi and Mariotti (2008) give an account of their data analysis of the actions of the teacher coming from teaching experiments (Falcade, 2006; Falcade et al., 2007). They identified a recurrent teacher action pattern and characterized four types of teacher actions in relation to the teachers' role in fostering the evolution of mathematical signs:

- "Ask to go back to the task
- Focalize on certain aspects of the use of the artefact
- Ask for a synthesis
- Synthesize" (Bartolini-Bussi & Mariotti, 2008, p. 775)

The first teacher action, *ask to go back to the task*, is used in situations when the teacher identifies the need for the students to recall their experience with the artefact for solving the task. Going back to the task formulation reactivates the artefact context and provides the opportunity for students' personal signs to emerge or re-emerge. The action is obviously fitting to use at the start of a classroom discussion; however, it is also relevant to use whenever there is a need for the students to recall the personal meanings that emerged in the artefact activity – for example, if the discussion has turned into unwanted paths. *Focalize on certain aspects of the use of the artefact* follows the first teacher action. After the discussion and reactivation of the artefact context has led to a rich net of signs, then there is a necessity for the teacher to focus the students' attention on pertinent aspects of the experience by emphasizing certain signs that are shared and highlight shared meanings related to these signs, which are coherent with the educational aim of the activity (Bartolini-Bussi & Mariotti, 2008).

The third teacher action, *ask to synthesize*, aims to make the students generalize and decontextualize the meanings that emerged in the specific experience with the artefact. The teacher does so by asking the students to condense and report what has been discussed in the classroom. This teacher action is expected to contribute to a shared environment where the teacher can introduce mathematical terminology. The final teacher action, *synthesize*, concerns the need for the teacher to provide the mathematically acceptable signs and formulations – stating the validity of the signs from a mathematical point of view, which the students can now consider as the final product of the evolution of the initial personal signs (Bartolini-Bussi & Mariotti, 2008).

2.2.3. Analyzing the process of semiotic mediation

If we consider the basic didactic hypothesis of the TSM, which is based on the internalization process and conveys that signs that are produced from activities with an artefact, are socially elaborated, then it is inferable that to analyze the internalization process, we may direct our attention to students' production and usage of signs in social interactions. Consequently, the evolution of student meanings may be analyzed by identifying the signs that students produce in social activities – such as verbal utterances, written signs or DGE actions – and interpreting the meanings underlying these signs. In particular, the evolution of meanings can be highlighted by identifying specific chains of signs – for example, chains of relations of signification (Bartolini-Bussi & Mariotti, 2008).

2.3. Revisiting the research questions in light of the theoretical constructs

The research questions that were formulated in Chapter 1 already refer to the reasoning competency. This reflects the fact that the choice of describing the mathematical aims of the project in relation to the KOM framework and its reasoning competency was already made at the start of the project. However, while the KOM framework is suitable for describing the educational goals in terms of the reasoning competency, it does not, as mentioned previously, allocate much attention to the context of digital tools in mathematics teaching and learning, even if we consider the aids and tools competency and the six mathematics teacher competencies. Therefore, the TSM is included to provide a frame focusing on the relationship between artefacts and cognitive development, as well as on the essential mediating role of the teacher. After the introduction of terminology from the TSM, we can add nuances to the research questions that were posed. The aim of the first research question, which was to uncover potentials of DGEs in relation to supporting students' development of reasoning competency, may now, in light of the TSM, be reformulated as the analysis of the semiotic potential of DGEs in relation to students' development of mathematical reasoning competency.

1. What is the semiotic potential of DGEs in relation to students' development of reasoning competency?

The second research question, which relates to the actual usage of the potentials in Danish lower secondary schools, can be rephrased as follows:

2. To what extent is the semiotic potential of DGEs in relation to reasoning competency currently exploited in Danish lower secondary schools?

At the time of writing papers I and II, which attempt to answer these first two questions, the project was not fully anchored in the TSM frame but rather in a state of exploring which theoretical constructs were suitable in relation to the overarching aim of developing DGE guidelines in relation to reasoning competency. Therefore, the above formulations may be put forward in hindsight, but in reality, the theoretical framing of the project was still in process as the first two research questions were being pursued. Hence, this is reflected in the research questions that are put forward in papers I and II. In fact, the anchoring of the project in the TSM started as a consequence of the review that is elaborated in the first paper.

The third and main research question of the project, which relates to the design of guidelines that utilize the potentials of DGEs in relation to reasoning competency, can be nuanced substantially by concepts from the TSM and may, in fact, be divided into several sub questions (a, b and c),

3. Which research-based guidelines feature in the design of teaching that utilizes the semiotic potential of DGEs in relation to reasoning competency?

Examining paper III, which focuses on analyzing students' signs that emerge in specific artefact tasks, it is quite directly linked to the theoretical frame provided by the TSM. The sub question that is answered in paper III is the following:

a. As students work on a designed dependency task, which type of signs emerge that are related to the use of the dragging tool and can be seen as evidence of students' awareness of the logical relationship between the geometrical properties in play? How can the unfolding of the semiotic potential from this case contribute to the formulation of guidelines?

Paper IV, instead, is centered on a specific part of KOM's reasoning competency – namely, the characteristic of being able to develop a heuristic argument into theoretical validation through a specific task design and didactic implementation. The following sub question is addressed:

b. How can students' conjecturing activities in DGE be combined with theoretical validation, to make theoretical validation a meaningful activity for the students?

Paper V focuses specifically on the role of the teacher and outlines the design that implements the four teacher actions from the TSM frame into guidelines for the teacher to manage the classroom discussion. The paper addresses the following sub question:

c. In what ways do the teaching guidelines support the teacher in holding classroom discussion; to what extent is the support consistent with our expectation; and based on the collected data, which revisions can we propose?

2.4. Other theoretical constructs

While reasoning competency and the TSM constitute the main overarching conceptual frameworks in this project, other theoretical constructs are drawn upon in order to formulate the guidelines and analyze them, to formulate task design principles and to shed light on the empirical data. These constructs include the following: the instrumental approach and the distinction between utilization schemes and instrumented techniques (Artigue, 2002; Drijvers et al., 2013; Guin & Trouche, 1999); van Hiele's (1986) model of five levels of mathematical thinking; the epistemic and pragmatic value of techniques that are performed with artefacts (Artigue, 2002); the distinction between spatiographical and theoretical levels of a figure (Laborde, 2005a); several DGE specific constructs, such as the distinction between robust and soft constructions (Healy, 2000; Laborde, 2005b) or different dragging modalities (Arzarello et al., 2002); the notion of proof as an explanation (de Villiers, 2007); and the design heuristic of Prediction–Observation–Explanation (White & Gunstone, 2014).

The constructs are elaborated ad hoc in the papers.

3. Methodological approaches, data collection, and data analysis

In this chapter, the methods that were applied in the project are described and justified, referring to the research questions. Furthermore, the data collection process of the project is elaborated upon, followed by ethical considerations that were taken. Finally, the data analysis approaches of the project are elaborated upon. Methods, data collection and data analysis are already described to a certain extent in the papers; however, in this chapter, an overview is also provided to highlight the alignment of the work and add details that were omitted in the papers because of space limitations of the journal and conference paper formats.

3.1. Methods

Answering the research questions necessitates considerations as to which methodological approaches should be included in the design of the study. One often distinguishes between quantitative and qualitative methods (Glatthorn, 1998), but one can also combine the two perspectives in a mixed methods design (Johnson et al., 2007), of which there are several typologies that may be suitable for various types of research goals (Creswell & Clark, 2011). Viewing the approaches from the point of view of the philosophy of science, then, on the one hand, the quantitative perspective is rooted in a view of knowledge that is related to positivism, where one collects and processes data that can be expressed in numbers. On the other hand, the qualitative perspective is linked to a phenomenological perception of knowledge, where concepts such as meaning and understanding are in focus (Glatthorn, 1998). The position within the philosophy of science that supports the mixed methods approach is American pragmatism – which argues for the use of mixed methods approaches, if it is an advantage in the research context, either because such an approach provides better answers to the questions one seeks or because it is easier to implement (Frederiksen, 2015). In this project, the mixed methods methodology is applied with a qualitative priority, which means that the quantitative method serves a secondary role, while greater emphasis is placed on the qualitative method (Cresswell & Clark, 2011, p. 65). The type of mixed method appropriated is related to what Creswell and Clark (2011) refer to as a multiphase design, also known as the sandwich design (Sandelowski, 2003). It occurs when

> an individual researcher or [a] team of investigators examines a problem or topic through an iteration of connected quantitative and qualitative studies that are sequentially aligned, with each new approach building on what was learned previously to address a central program objective. (Creswell & Clark, 2011, p. 100)

As previously mentioned, the third research question constitutes the central objective of the project. However, accomplishing the central objective involves building on the results identified in the work produced from the studies that address the first and second research questions (see Figure 3).

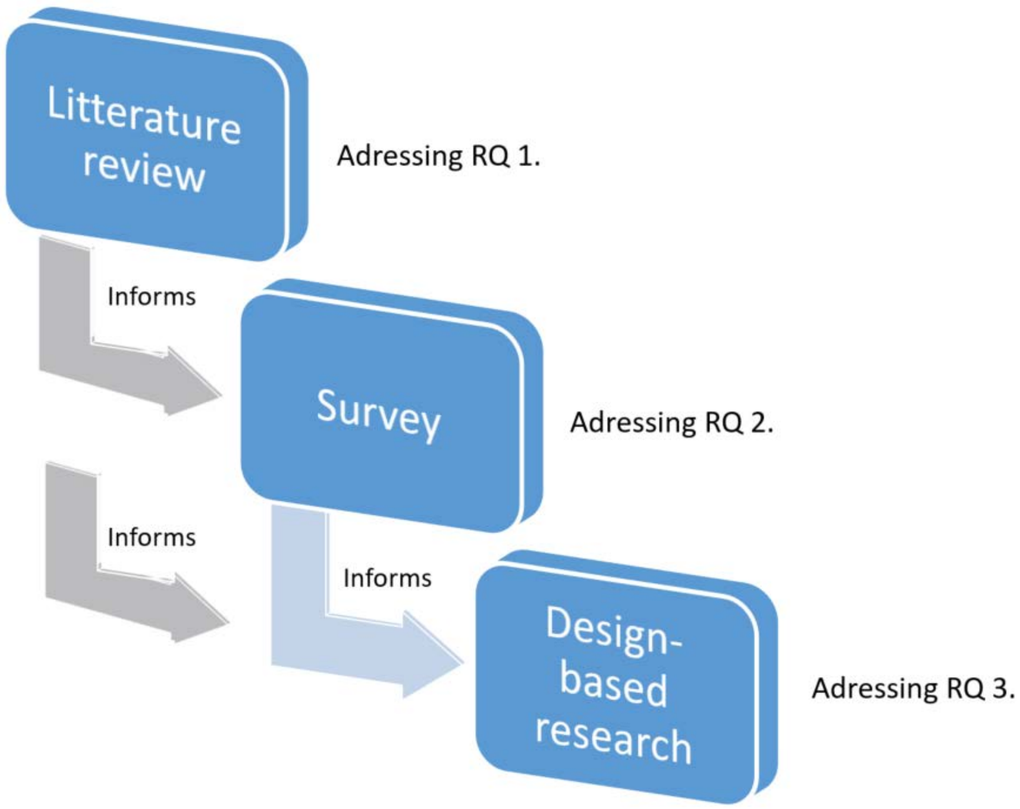

Figure 3. The connected quantitative and qualitative studies in the project

The project is methodologically branched into three parts, corresponding to the threefold research question. Each branch is elaborated upon in the following sections.

3.1.1. The literature review

The first research question involved identifying potentials and challenges with DGEs in relation to teaching that aims to support students' development of reasoning competency. A review of previous DGE literature was deemed to be a suitable method to give insights into what these potentials may comprise.

The review was methodologically anchored in the hermeneutic framework for literature reviewing (Boell & Cecez-Kecmanovic, 2010, 2014). A characteristic of the hermeneutic approach is that initial research questions and keywords do not lock in the scope of the review from the outset. Instead, they function precisely as that – initial research questions and keywords, which can be refined or changed if it is found meaningful to do so along the way. Since reading literature obtained by an initial search can expand one's understanding of the field, this may inform new keywords and revised research questions, giving rise to new searches and reading. Through repetition of this process, a better and deeper understanding of the field may be accumulated. Boell and Cecez-Kecmanovic (2010) question the suitableness of systematic review strategies for Ph.D. students:

> A deeper understanding of the research problem is gained as the literature review progresses, with the researcher becoming more aware of what questions are most relevant or pressing. Systematic review strategies are therefore ill equipped to address research that cannot be precisely formulated in the form of closed questions before starting the review process. Claims by proponents of systematic reviews that this method is suitable for research students undertaking a PhD (Kitchenham, 2004) are therefore open to question. (Boell & Cecez-Kecmanovic, 2010, p. 131)

This review method was chosen, following the advice of the authors, since I was uncertain of the conceptualization used in literature in the field of DGEs and reasoning competency, and of the scope of the literature. Therefore, the flexible and accumulative process offered by this method seemed conducive.

Applying the method involved operationalizing the main characteristics of reasoning competency so that it was possible to conduct searches in databases to locate literature that is related to reasoning competency. Since the specific KOM meaning of reasoning competency is not widely used in mathematics education research, it was not feasible to search by directly using reasoning competency as the search word. Instead, several search words were used, initially “reasoning,” “conjecture,” “justify” and “proof” to represent reasoning competency, and these were combined with synonyms of a DGE, “dynamic geometry,” “geometry software,” “geometry technology” and “interactive geometry,” using AND/OR operators to search the MathEduc[3] and ERIC[4] databases for relevant publications. The initial search was followed by citation tracking, as well as new searches that included the search words “counterexample,” “argumentation” and “heuristic proof” and searches focusing on specific theoretical constructs, specifically “instrumental genesis,” “semiotic mediation” and “Hiele.”

The constituents of reasoning competency were also important in the *sorting, selection, critical assessment* and *argument development* phases of the method (Boell & Cecez-Kecmanovic, 2010) because they acted as the lens used to sort and analyze the acquired literature to extrapolate which affordances of DGEs can be considered as potentials when the educational aim is to foster students’

[3] https://www.zentralblatt-math.org/matheduc/

[4] https://eric.ed.gov/

development of reasoning competency. Note that the notion of "potential" is used in this context to refer to affordances of DGEs, which are novel in comparison to affordances that are provided by other common mathematics education resources – in particular, in comparison to pencil, paper, ruler and compass geometry.

The results of this work, which is described in the first paper (Højsted, 2020a), forms the basis of the following work on the second and third research questions. There is an overlap between the review and the DBR process. In fact, a priori guidelines are developed in this first paper, which connects it quite clearly to the third research question. The aim of the review was, therefore, not only to identify potentials and challenges with DGE but also to form the basis needed to develop a priori guidelines for teaching that utilize these potentials to foster students' development of reasoning competency (described in further detail in section 3.1.3).

3.1.2. The survey

Answering the second research question requires quantitative data from across the country, which a survey can provide. This quantitative method was therefore applied in the form of a web-based questionnaire that was developed and administered using SurveyXact[5] to Danish lower secondary school mathematics teachers.

The survey development process involved formulating questions that could provide insights as to what extent the DGE potentials (dragging, feedback, measuring and tracing) identified in the review synthesis (Højsted, 2020a) are utilized in the teachers' mathematics classrooms. However, it could not be expected that the lower secondary school teachers were familiar with the DGE terminology predominantly used in scientific discourse (Hansen & Andersen, 2009). The task at hand was therefore to consider which questions could cover what I wanted to know while being formulated in a manner that was expected to resonate with the discourse used by a typical lower secondary school mathematics teacher. This was not a trivial task – in particular, with regards to certain DGE elements (e.g., dependent and nondependent objects, invariants) that were difficult to present using everyday language while maintaining a clear and concise formulation. However, some explanatory texts were used prior to certain questions, which may have alleviated possible misunderstandings. Additionally,

[5] https://www.surveyxact.com/

a short video was shown to the respondents to explain what was meant by “free objects” and “locked objects” in GeoGebra prior to questions that involved these concepts (see Figure 4).

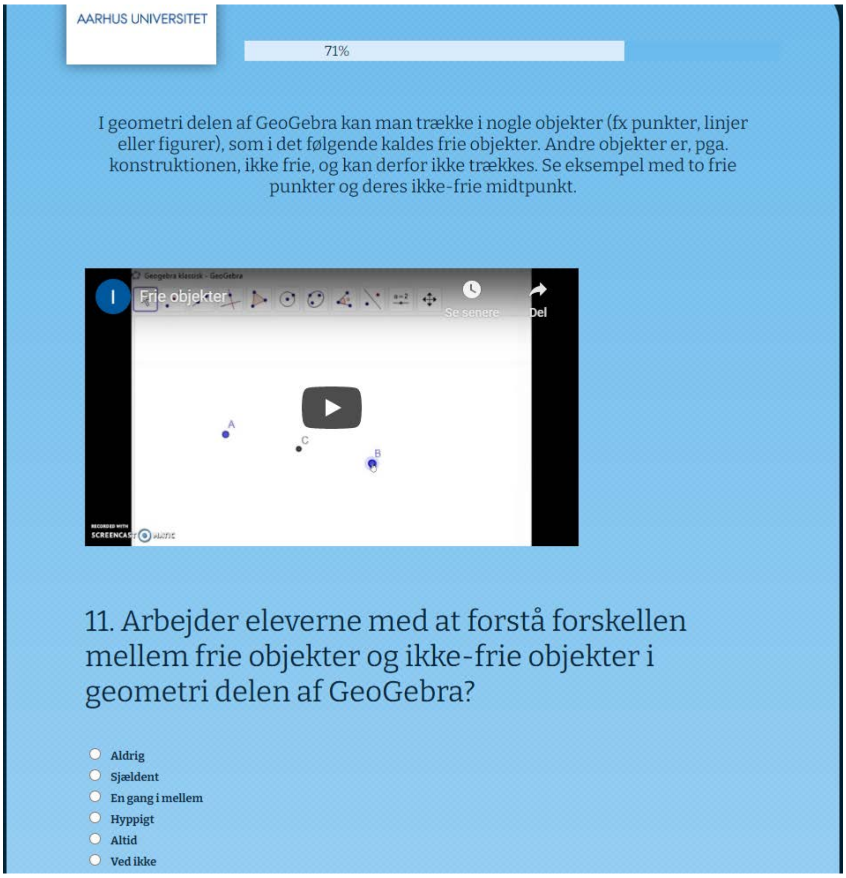

Figure 4. Snapshot from the questionnaire in which a short explanatory video is embedded

Nineteen multiple-choice questions were formulated, in which the teachers were asked to assess how frequently certain tools in GeoGebra were used in particular ways and to solve specific tasks. Since GeoGebra is by far the most popular DGE in Danish lower secondary schools, I chose to formulate questions concerning GeoGebra usage instead of DGE usage, thereby making the questions more recognizable and clearer for the participants. The multiple-choice questions were assigned a five-point Likert scale spanning a spectrum of frequency of usage from “always” to “never,” as well as an answer option of “don’t know” (Allen & Seaman, 2007). In addition to the multiple-choice questions, four open-ended questions, as well as some background questions, were developed. The purpose of the open-ended questions was to get detailed and rich answers that would allow for a deeper and more nuanced analysis concerning certain questions regarding GeoGebra utilization – for example to get the teachers’ points of view of what they consider to be important potentials of GeoGebra. The background questions allowed for discerning the respondents according to certain categories (e.g., which grade they primarily teach).

Developing a questionnaire involves considering which types of questions may be fruitful to acquire the information one seeks. Hansen and Andersen (2009) argue that a question often has three dimensions: (i) a dimension concerning the content, which can either relate to something factual, something cognitive, or revolve around attitudes to something and an evaluation of something; (ii) a time dimension – a question enquires about the past, the present or the future; and (iii) a response dimension, where there are either preconstructed response categories or open response options.

From this perspective, starting with the content dimension, all but two of the questions in the questionnaire are *factual* questions (Hansen & Andersen, p. 105) – for example, asking the teachers if their students' task work includes constructing "robust figures." The remaining two questions can be characterized as *attitude* questions of an *evaluative* nature (Hansen & Andersen, p. 106), as the teachers were prompted to evaluate the potentials of GeoGebra. In the time dimension, it was chosen, for the sake of clarity, that all the questions should enquire about the present time, using phrases such as "Do students work on investigating figures …?" In relation to the response dimension, the questionnaire contained, as mentioned, mainly closed preconstructed response categories but also four open-ended questions.

Certain questions were posed to unveil the types of tasks that the teachers give to their students. The rationale for posing these questions was, on the one hand, to investigate to what extent the hypothesis that GeoGebra primarily serves as a substitute to paper and pencil. On the other hand, the answers to these questions could also indirectly reveal if the potentials were utilized – for example, if the students primarily work on tasks made for the paper and pencil environment, in which dragging is not possible.

Before initiating the survey, it was decided to test it through a small pilot survey. Pilot studies can have many functions (Schreiber, 2008, pp. 624–625), but in this case, the purpose was to investigate whether the respondents could understand the questions as it was intended, as well as to get an initial impression of what sort of answers could be expected in the open-ended questions. The pilot group consisted of 16 master's students studying mathematics education at Aarhus University. Approximately two-thirds of the pilot group students had a background as primary and lower secondary school mathematics teachers (grades 0–10), while the remaining third had a background in upper secondary school mathematics teaching (grades 11–13). Therefore, the pilot group was reasonably well suited in terms of their background, and in practice, the pilot proved useful, leading to minor revisions of the questionnaire.

The methods of distribution of the questionnaire were twofold. On the one hand, a link for self-enrolment to the questionnaire was administered through the email list of the comprehensive Danish mathematics counselor network,[6] which reaches mathematics counselors and resource persons in primary and lower secondary schools in 97 of the 98 municipalities in Denmark. The link was accompanied with a text containing information about the survey and asking the mathematics counselors to share the link with their colleagues. On the other hand, a link for self-enrolment was posted on two popular Facebook groups for Danish lower secondary mathematics teachers – namely, "We Who Teach in Lower Secondary Schools"[7] and "GeoGebra Hangouts."[8]

Although web-based surveys have become a predominant method in educational research (Fan & Yan, 2010; Saleh & Bista, 2017), researchers point to the issue of low response rate and completion in web-based surveys, with some estimates showing it to be 11% lower than other survey modes (Fan & Yan, 2010).

Certain steps were taken concerning design and language (Fan & Yan, 2010) in order to compensate for the low response rates expected with the online survey format. These steps include that the survey was structured so that easier questions came first in order to get respondents started and that the questions were formulated as clearly and concisely as possible, with video and explanatory texts added. Another important issue related to response rates are assurances of privacy and confidentiality (Saleh & Bista, 2017). The first page of the survey contained an introduction assuring the respondents of their anonymity and explaining exactly which information about them would be collected and for what purpose. A consent box was also added, which the participants had to tick to advance. A further step was to add a monetary incentive for the respondents, which increases participant response and completion (Göritz, 2010; Saleh & Bista, 2017). The incentive took form as a lottery with a single prize of DKK 4000 for the winner. To participate in the lottery, the respondents were required to complete the questionnaire and fill in their names and email addresses in a separate box. The names

[6] https://phabsalon.dk/matnet/om-dmn/

[7] https://www.facebook.com/groups/1827579010797172/

[8] https://www.facebook.com/groups/geogebrahangouts/

and email addresses were needed to identify and contact the winner, but it also provided the possibility of checking for double entries.

The study and results are described and analyzed in the second paper (Højsted, 2020b). The analysis centered on certain questions for the sake of a focused dissemination, yet the conclusions provide an accurate account of the whole dataset. The questionnaire in its entirety can be viewed in appendix A.

3.1.3. DBR

To address the third research question, the DBR methodology was applied, adhering to the principles and advice laid out by Bakker and van Eerde (2015). The approach was chosen because DBR serves "an explanatory and advisory aim – namely, to give theoretical insights into how particular ways of teaching and learning can be promoted" (Bakker & van Eerde, 2015, p. 431). Seeing as the objective at hand in the third research question was to develop guidelines for teaching that utilizes potentials of DGEs in relation to supporting students' development of reasoning competency, it was expected that the approach could contribute to that end – specifically, by developing theoretical insights into how teaching with DGEs for this particular aim may be approached. Since such teaching was not expected to be found in the current praxis, it was necessary to design such activity to be able to study it.

The DBR approach has gained increased attention as a research methodology in mathematics education since the early 1990s (Artigue, 1994, 2009; Brousseau & Balacheff, 1997; Freudenthal, 1991; The Design-Based Research Collective, 2003; Wittmann, 1995). The main constituents of DBR involve the design of educational materials such as computer tools or educational activities. The approach usually comprises of cycles of three phases each: (i) preparation and design of the educational materials, (ii) testing of the educational materials in a teaching experiment and (iii) a retrospective analysis of the collected data from the teaching experiment. The design process is anchored in theory, and hypotheses about the expected outcome are described a priori. Through iterations of testing and redesigning the educational materials, new insights may be gained concerning the design, and new theory development may arise. Hence, the design of learning environments is interwoven with theory testing and development. Bakker and van Eerde (2015) argue that the strength of DBR lies in its potential of connecting educational practice and theory since it is centered on theory development concerning domain-specific learning, as well as the steps designed to guide that learning. Therefore, DBR generates beneficial resources (e.g., educational materials) as well as systematic insights into the possible utilization of these resources in education (Bakker & van Eerde, 2015, p. 430; McKenney & Reeves, 2012; Van den Akker et al., 2006).

Next, the main characteristics of DBR are introduced, as well as what is understood by theory and theory development, before the manner in which the approach was applied in the project is presented.

Characteristics of DBR and the role of theory

Bakker and van Eerde (2015, pp. 452–453) summarize five of the main characteristics of DBR: (1) the purpose of DBR is to "develop theories about learning and the means that are designed to support that learning" (Bakker & van Eerde, 2015, p. 452). (2) DBR is an interventionist approach. DBR interventions are usually situated in natural learning situations in schools, which can provide better ecological validity compared to controlled laboratory situations. (3) DBR has prospective and reflective components. The prospective part refers to the hypothesized learning, which is theorized before the teaching experiment, while the reflective part refers to the analysis of what is actually observed in the teaching experiment and how it matches conjectures made beforehand. (4) The nature of DBR is cyclical, with each cycle consisting of a design phase, a teaching experiment and a retrospective analysis that then feeds a new design phase. (5) The theory under development is usually humble and specific (see hypothetical learning trajectories below) since it is developed for a specific domain. However, it may contain elements of generality that may be applicable to other contexts.

The role of theory is central to DBR, and it is the element that distinguishes it from certain other approaches, such as action research (Bakker & van Eerde, 2015). However, the notion of theory is somewhat ambiguous in mathematics education research, with far ranging research perspectives being utilized (Niss, 2007) to understand and explain the complex and multifaceted phenomena involved in mathematics learning and teaching (Bikner-Ahsbahs & Prediger, 2010). The role of theory in DBR that is used in educational research can be categorized into five types of theory (diSessa & Cobb, 2004). These types of theories, which are described below, vary in terms of generality, from grand theories to context-specific theories.

- "Grand theories (e.g., Piaget's phases of intellectual development; Skinner's behaviorism)
- Orienting frameworks (e.g., constructivism, semiotics, sociocultural theories)
- Frameworks for action (e.g., designing for learning, Realistic Mathematics Education)
- Domain-specific theories (e.g., how to teach density or sampling)

- Hypothetical Learning Trajectories (Simon, 1995) or didactical scenarios (Lijnse, 1995) formulated for specific teaching experiments." (Bakker & van Eerde, 2015, p. 437)

DBR projects may utilize theories on different levels in the same project, for example grand theories, frameworks for action and a specific hypothetical learning trajectory. It is, in fact, recommended to do so; however, it is necessary to ensure alignment between the different types of theories and the research design (Bakker & van Eerde, 2015).

The hypothetical learning trajectory is a central instrument in DBR. It is used to describe the learning aim that specifies the path to be taken in a particular teaching experiment. It also defines the learning activities expected to foster the specified learning aim, as well as hypotheses about the expected evolution of students' understanding as a consequence of the suggested learning activities. The basis for theory development lies in the coordination between the hypothetical learning trajectory and the empirical results gathered in the teaching experiment (Bakker & van Eerde, 2015).

DBR applied in the project

The main product developed in this project is guidelines for the design of educational activities that utilize potentials of DGEs to support students' development of reasoning competency. In the frame of DBR, the guidelines take the form of a hypothetical learning trajectory that is developed to reach the specified project goal. The term "guidelines" was chosen to represent hypothetical learning trajectory/didactical scenarios, which are commonly used terms in DBR (Bakker & van Eerde, 2015).

Using the guidelines, educational activities are designed that serve the role of testing the guidelines to empirically qualify and revise them.

(i) Preparation and design

As mentioned previously, there was an overlap between the review and the DBR part of the project because the development of the guidelines started with the review. As a part of the preparatory phase, the aim of the review was precisely to develop a priori guidelines based on an analysis of DGE literature. To do so, it was necessary to identify potentials and challenges with DGEs when the educational goal is to foster students' development of reasoning competency. It was also necessary, in the review process, to unveil which dimensions the guidelines should entail and which theoretical constructs were useful for this aim. The guidelines are aligned, theoretically, to the TSM model (Bartolini-Bussi & Mariotti, 2008), which is utilized in the project as a framework for action. The guidelines include the analysis of the semiotic potential of DGEs in relation to students' development of reasoning competency. The TSM is in itself anchored in the orienting frameworks of semiotics and sociocultural theories (Vygotsky, 1934/1978).

Results from the survey provided updated insights into DGE utilization in lower secondary schools, which was also valuable in the design process. These results include the indications that a DGE was used as a substitute to solve paper and pencil tasks and the lack of focus on developing students' awareness of the dependencies that are mediated as invariants during dragging – for example, being aware of the difference between free and locked objects in DGEs (see more in Højsted, 2020b). Consequently, tasks were designed aiming to foster this awareness.

In addition to the review and the survey, the preparatory phase involved analyzing several materials that were relevant for the design. These included task design literature, textbooks and teaching materials, online materials such as GeoGebra applets,[9] Danish lower secondary school mathematics exam items and the existing curriculum in order to investigate how aims concerning both reasoning competency and DGE are elaborated.

On the basis of this work, a didactic sequence was designed. The sequence included the design of 15 tasks adhering to the a priori guidelines described in Højsted (2020a), as well as other considerations that were taken into account. The considerations comprised not only criteria for task design but also limitations that were applied, which had nothing to do with the learning aims but were consequences of pragmatic research limitations. For example, it was necessary to keep the workload manageable in terms of data collection and, not least, in terms of data analysis and dissemination. It was decided to aim for a three-week didactic sequence. The tasks were designed referring directly to GeoGebra since it is the commonly used DGE in lower secondary schools in Denmark.

Below is a summary of the overarching criteria and principles that guided the design of the tasks in the didactic sequence:

- The tasks of the didactic sequence referred in its entirety to most of the levels described in the guidelines (Højsted, 2020a).
- At least one of the four potentials (dragging, feedback, tracing and measuring) (Højsted, 2020a) was utilized in each of the tasks of the sequence.

[9] https://www.geogebra.org/materials

- Most of the five types of tasks identified in the review (Højsted, 2020a) were included in different variations.
- The design heuristic of Predict–Observe–Explain (White & Gunstone, 2014, pp. 44–65) was incorporated in the task design (see more in Højsted & Mariotti, 2020a).
- The design invention of the toolbox puzzle approach (Højsted, 2020c) was used in the tasks in which the students were to verify their conjectures.
- Certain task design choices were taken that are not specific to DGE tasks but rather serve the intention of mobilizing characteristics of reasoning competency in a generic manner. For example, it was chosen and incorporated into the tasks that the students would work in pairs, and in some subquestions, they were asked to describe how they expected DGE constructions to behave and to justify their assumptions to each other. The choice reflects the aim of fostering the students' abilities to communicate and put forward justifications for mathematical claims, which is an element of reasoning competency. The choice also served the research purpose of providing a "window" into the students' thinking processes as they were solving the tasks.
- The design attempted to create a connection between the tasks so that insights gained from one task were utilized in subsequent tasks.

Several other criteria were considered; however, during the design phase, it became evident that it would be difficult to adhere to too many criteria. Therefore, compromises and decisions had to be made, favoring certain criteria, while others had to be relegated. One such decision that is worth mentioning concerns the openness and explorative nature of the tasks. The initial idea was to have quite open tasks without too much guidance (Olsson & Granberg, 2019) so that the students could get engaged in a mode of problem solving and use the possibilities of exploration in DGEs to get answers. However, solving such tasks takes time, and since there were many types of tasks that needed to be tested according to other criteria, it was decided to go with guided explorations as one of the pragmatic solutions to the problem of too many considerations and a short implementation period. A more fine-grained account of the task design rationale for some tasks is presented in the structure of objective, hypothesis and choices in paper III (Højsted & Mariotti, 2020a).

The tasks used in the didactic sequence from iteration 2 are available in appendix B. Other materials were designed as a part of the didactic sequence. These include a document to the teacher concerning the aim of the tasks (see appendix C), a lesson plan, the desired students' answers to each task and

PowerPoint slides for the teacher to use during classroom discussions. Following the TSM model, the lesson plan included iterations of working on tasks, followed up by teacher-led classroom discussions.

Between the second and the third design iterations (see the next section), teacher guidelines were developed to support the teacher in facilitating the classroom discussions. The teacher guidelines were developed based on the TSM frame adapted in an attempt to make it understandable for the teacher. The teacher guidelines included a description of the didactic cycle, a description of four general teacher actions to be used in the classroom discussions, a scheme containing descriptions of the mathematical aim of each task, the personal signs that could be expected to emerge from the activity (hypothetical and based on previous iterations) and possible teacher actions as a response to the signs that are expected to emerge (see appendix D).

Since the term "guidelines" is used in relation to two products now, it is worth distinguishing between the different usages, to avoid confusion. The overall research aim, in accordance with the third research question, is to develop guidelines for the design of didactic sequences utilizing potentials of DGEs to foster students' development of reasoning. The above mentioned teacher guidelines, which were developed to support the teacher in managing the classroom discussions, are only one component of the didactic sequence design.

(ii) Testing of the educational materials in a teaching experiment

The didactic sequence was tested, analyzed and redesigned in three design cycles, including the pilot study (see Figure 5).

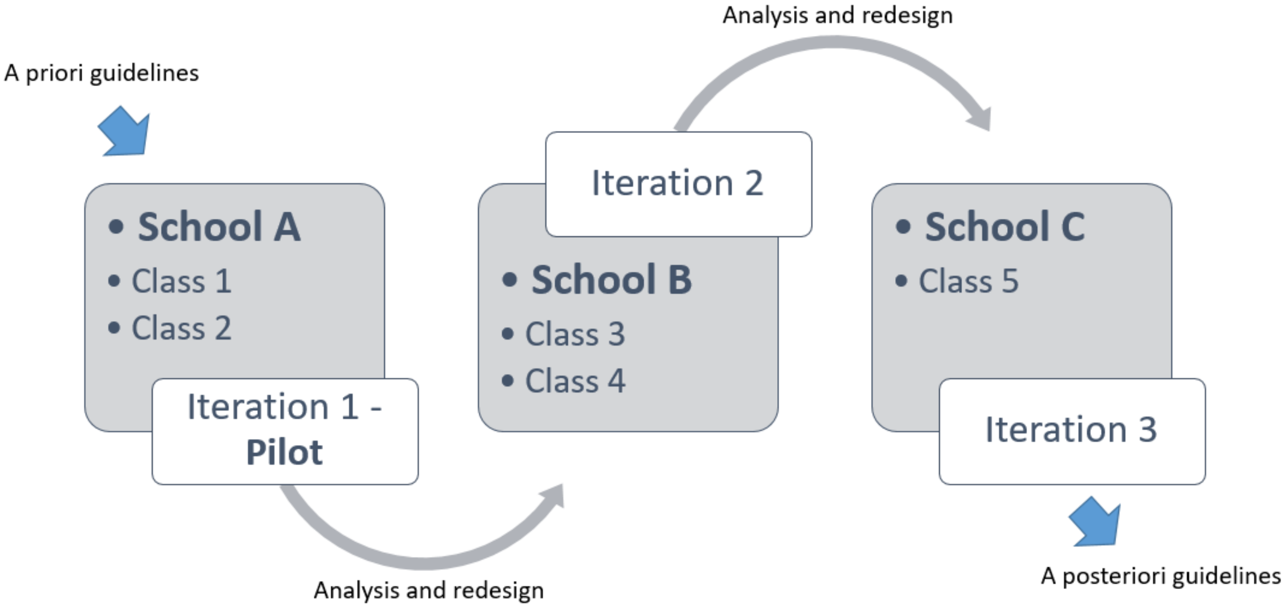

Figure 5. An overview of the schools and classes involved in the iterations of testing

The schools in the project had participated in previous experiments that are unrelated to this project; hence, a line of communication was available. Five teachers showed interest and agreed to be a part of the project: two teachers from school A, two from school B and one from school C. Each teacher participated with one class.

Prior to the start of each intervention, meetings were held with the teachers to go through and discuss the teaching material in order to develop a shared understanding of the aim of the sequence and get the teachers' views on the material, including their suggestions for improvements. The discussions also served as an opportunity to unveil the experience and preconditions of the teachers and their students in relation to GeoGebra proficiency and to reasoning competency. In the third iteration, the teacher guidelines were also discussed prior to the intervention. In the ongoing intervention, short meetings were held with the teacher after each lesson to discuss how it went and prepare for the next lesson.

I, as the researcher, assumed a teacher-supporting role during the didactic sequence in classes 1–4, especially during the start-up phase. The involvement varied in the classes based on the confidence/requirements of the teachers for support. In class 5, the teacher carried out all the teaching, while I remained as an observer for the entirety of the sequence, except for an introduction to the project at the start. This was chosen because in the third iteration, the research focus was particularly on the role of the teacher, with the aim of testing how the teacher managed the classroom discussion in light of the teacher guidelines developed for this purpose.

The experiment with class 1 was a bit of an outlier compared to all the other classes due to circumstances surrounding the teacher who was responsible for this class. The teacher, contrary to expectations, did not teach eighth grade at the time of the intervention, and therefore, class 1 had to be a ninth-grade class. The class also had other activities planned, which meant that they could only participate in the project for two weeks instead of the planned three weeks. Nonetheless, it was still deemed valuable to collect first impressions from this class, particularly in relation to the testing of data collection equipment and setup.

Class 2 was an eighth-grade class, as were the remaining classes in the experiment. The experiment in class 2 ran as an immediate extension of the first class; therefore, apart from minor adjustments, the material used was the same in both classes. Similarly, the didactic sequence material was nearly identical in classes 3 and 4. Hence, the majority of the design changes occurred between iterations.

Data were collected in the form of screencast of students' work in the DGE; videos of three chosen groups from each class (only two groups in class 5), which were chosen in collaboration with the teacher so that the groups represented a spectrum with regards to mathematical background from low-achieving students to high-achieving students; audio recordings of the chosen groups (in the second and third iterations); interviews with the teachers; students' written products; and video recordings of the whole class.

(iii) Retrospective analysis of the collected data from the teaching experiment

Retrospective analysis was performed after each iteration, which included task-oriented analysis (Bakker & van Eerde, 2015). The analysis of data aimed to identify to what extent the outcome of the educational activities lived up to the expected learning outcome that was hypothesized beforehand and, on the basis of this analysis, how the educational activities should be redesigned and, consequently, how the guidelines, on which the educational activities were based, should be revised.

During the research project, it became clear that it was necessary to sharpen the analytical focus in order to qualify the outcome of the research and keep the workload manageable. The guidelines and the resulting didactic sequence contained many layers of learning aims. Instead of developing a coarse-grained analysis of each of these layers in the whole didactic sequence, it was deemed more useful to make a fine-grained analysis of selected aspects that might provide insights to the research community. Therefore, even though the design involved elements from each level of the guidelines, the analysis centered on certain aspects of the design and, hence, on certain elements of the guidelines. A deciding factor for the choice of foci was that the results from the survey (Højsted, 2020b) indicated a lack of focus in Danish lower secondary schools on students' awareness of free and locked objects in DGEs, which is a part of step one in the guidelines. In addition, as described in paper I (Højsted, 2020a), the dimensions of task design and the role of the teacher are underdeveloped. Consequently, the choice was made to focus the research analytically by zooming in on level 1 in the guidelines and, in that context, on task design and the role of the teacher anchored in the TSM, in addition, to zooming in on the toolbox puzzle approach that was designed to support the students in verifying their conjectures.

The outcome of this analysis is reported in paper III (Højsted & Mariotti, 2020a), zooming in on level 1 in the guidelines; in paper IV (Højsted, 2020c), zooming in on the toolbox puzzle approach; and in paper V (Højsted & Mariotti, 2020b), zooming in on the development of guidelines for the teacher.

3.2. Data collection and analysis

As mentioned already, data were collected in different forms in the three parts of the project and analyzed in different ways.

The data analyzed in the review comprised 136 peer-reviewed publications. The characteristics of reasoning competency had the crucial function of acting as the analytical lens in the sorting and selection of the data to identify the potentials of DGEs in relation to reasoning competency and inform the development of a priori guidelines.

The data from the questionnaire consisted of answers from 700 teachers teaching in primary and lower secondary schools. The data were sorted for the analysis so that only full answers were included, and only answers from teachers that teach in lower secondary schools, which was the age group of interest. Therefore, paper II deals with data from 220 respondents that fit those criteria. Certain questions were chosen for presentation in the paper II analysis in the form of frequency tables, as well as a categorization of the teachers' comments into an open-ended question. The results of the analysis presented was chosen so that it would reflect the results from the survey as a whole – that is, adding further results would not contribute to the synthesizing result or revise the conclusion. An overview of all of the questions and the full 700 answers is provided in appendix A.

As mentioned previously, data from the DBR part of the project was collected in the form of screencast recordings of the students' work on the computer, external video recordings of three chosen groups from each class (only two groups in class 5), audio recordings of chosen groups in the second and third iterations, interviews with the teachers, students' written products and videos of classroom discussions.

The screencast software chosen was OBS Studio (https://obsproject.com/). The software allowed for the inclusion of a webcam recording as a part of the screen recording, which made it easier to identify who was talking. This function was not used initially but proved to be a necessary step that became evident after the pilot study. The data collected in the pilot study was deeply affected by sound problems. It was difficult to decipher what the students were saying because of overlapping voices. The webcam video helped with this matter. Another remedy used from iteration 2 onward was to relocate the chosen groups for external video in the classroom so that there would not be too much noise from neighboring classmates. A third and efficient remedy was to secure extra audio recordings of the chosen groups using quality audio recorders. The quality of the data from the pilot study was too poor to be used directly in the analysis and therefore served primarily as a testing ground for the second iteration in terms of data collection.

The data analysis in paper III used screencast data, external video, transcripts from audio recordings and students written products from class 3 and 4 (mainly 3) in the second iteration. These data were analyzed in the TSM frame – searching for signs that emerged in the artefact activity and interpreting the underlying meaning of the students' signs and comparing the results to the expected outcome hypothesized a priori from the task design point of view, leading to an evaluation of the task design principles.

Data analysis in paper IV used data from class 3 and 4 (mainly 4) in the second iteration. Screencast, external video, audio recordings and written products were used in the analysis; however, due to the eight-page limit of conference papers, which paper IV was restrained to, only data from external video, written products and transcripts of audio recordings were presented in the paper. Due to the same reason of space limitations, I decided to omit the fact that the study was theoretically anchored in the TSM, both concerning the (semiotic) potential of DGEs in relation to conjectures and proof and the presentation of the results of analysis of two students' work on the toolbox puzzle approach (the unfolding of the semiotic potential).

The data presented in paper V stem from class 5 in the third iteration. Data were collected in the form of videos of the whole class and selected groups, screencast recordings of all groups, audio recordings and written products from the students. Data were also collected by interviewing the teacher before and after the teaching experiment and after each teaching session through a semi-structured interview. In the paper, only transcripts from the classroom discussion and the teacher interview after the first lesson were presented and analyzed.

Table 1 below provides an overview of the methods applied, data collected and analysis conducted in the papers of the project.

Paper	Method/type of research	Data collected	Data analysis
1.	Hermeneutic framework for literature reviewing	136 peer-reviewed publications.	Identifying the potentials of DGEs in relation to reasoning competency. Informing the development of a priori guidelines.
2.	Quantitative study – questionnaire	Answers from 220 Danish lower secondary school teachers	Identifying the utilization of DGE potentials. Using frequency tables. Categorizing comments.
3.	Qualitative study – design-based research	Videos, screencast, transcripts from audio recordings, written products coming from iteration 2, class 3	Analyzing the unfolding of the semiotic potential. Interpreting students' emerging signs. Evaluating task design principles related to a dependency task.
4.	Qualitative study – design-based research	Videos, screencast, transcripts from audio recordings, written products coming from iteration 2, class 4	Analyzing the unfolding of the semiotic potential. Interpreting students' emerging signs to evaluate the task design related to the toolbox approach.
5	Qualitative study – design-based research	Videos, screencast, transcripts from audio recordings, written products, and semi-	Analyzing the unfolding of the semiotic potential. Analyzing the teachers' management of the classroom discussion to evaluate the

		structured interview coming from iteration 3, class 5	designed teaching guidelines. Analyzing interview transcripts.

Table 1. An overview of methods and data in the project

3.2.1 Ethical considerations

An important part of research ethics concerns the treatment of the people involved in the research (Vetenskapsrådet, 2017). There were participants in the data collection process of the survey and in the DBR part, which clearly required ethical considerations. The purpose and scope of the data collection, the duration of the storage of the data and the way it would be used in anonymized form was explained to all participants prior to their participation, and their consent to these conditions was required for them to participate. In the web-based survey, these conditions were explained on the first page of the survey, and it contained a box that the respondents had to tick to participate. Since there was a lottery, the respondents that wanted to participate in the lottery had to enter their names and their email addresses, which made the data particularly sensitive; however, this part of the data was deleted as soon as the lottery winnings were transferred to the lottery winner.

In the DBR iterations, which involved video recordings, consent forms were handed to the students from each class, which explained the purpose and scope of the data collection, the duration of storage of the data and the way it would be used (in anonymized form for research purposes only). The students and their parents were required to sign the consent form for the students to participate in the project. Informed consent was also collected from the teachers.

While undertaking this project, I have, to the best of my abilities, tried to live up to the principle of honesty "developing, undertaking, reviewing, reporting and communicating research in a transparent, fair, full and unbiased way" (ALLEA, 2017, p. 4) and emphasized other national and international ethical standards: credibility, integrity, responsiveness, transparency and accountability (ALLEA, 2017; Ministry of Higher Education and Science, 2014).

4. Results

This chapter outlines the results of the project, referring to the papers and briefly describing the results that each paper provided. The chapter concludes with a section that summarizes the results in a list. I suggest reading the papers in their entirety before or after this chapter if they have not been read already.

4.1. Review to identify potentials and develop a priori guidelines

Paper I: *Guidelines for Utilizing Affordances of Dynamic Geometry Environments to Support Development of Reasoning Competency* (Højsted, 2020a).

The purpose of the paper was, on the one hand, to unveil potentials of DGEs in relation to reasoning competency. Potentials in this context refer to affordances of DGEs that are not available with other common mathematics education tools – in particular, contrasting DGE with paper, pencil, ruler and compass geometry. Four potentials were identified: dragging, feedback, measuring and tracing. On the other hand, the aim was to develop a priori guidelines for the design of didactic sequences that utilize these potentials and to discover which dimensions the guidelines should entail. At the time of writing this paper, the project was not yet theoretically anchored in the TSM. In fact, the review phase was used to explore and decide which theoretical constructs were suitable to answer the overarching research questions of the project and contribute to the development of guidelines. The instrumental approach (Artigue, 2002), the TSM (Bartolini-Bussi & Mariotti, 2008), and the van Hiele model of levels (van Hiele, 1986) were used. The utilization of the potentials was described in three dimensions of guidelines: students' cognition, task design and role of the teacher. More specifically, the guidelines comprise a learning trajectory description of cognitive progression that is elaborated in terms of utilization schemes and instrumented techniques, five types of task design that are expected to foster this progression and the role of the teacher in facilitating this process. Finally, the guidelines convey the expected mobilization of reasoning competency, which is a consequence of this work.

4.2. The questionnaire

Paper II: *Teachers Reporting on Dynamic Geometry Utilization Related to Reasoning Competency in Danish Lower Secondary School* (Højsted, 2020b).

The aim of the paper was to unveil to what extent the four potentials are utilized in Danish lower secondary schools and give insights into the current DGE teaching practice, which could be used in the design of the didactic sequence.

Analysis of the data indicates that dragging and measuring are used to some extent, feedback is used less and tracing is almost not used at all. There are signs that the paper and pencil environment has been substituted with DGEs to solve tasks that were originally designed for paper and pencil, meaning that the new potentialities offered by DGEs are not used. The teacher most often highlights the pragmatic value of techniques. The results also indicate that the awareness of locked and free objects in GeoGebra is not a particular focal point – which, in the guidelines, is placed at level 1. Since locked and free objects are the manifestations of the geometrical properties of figures, which are mediated perceptually in DGEs during dragging as a consequence of both Euclidean theory and software design choices, they are important to be understood in order to be able to interpret what happens on screen. It was decided to focus on this finding concerning dependency relations in DGEs in the design of the initial tasks of the didactic sequence and in the consequent analysis, which is presented in paper III.

There are methodological and analytical issues in the study, one being that the data does not constitute a representative sample of lower secondary school mathematics teachers. In fact, as explained in the paper, the respondents may be considered as "super users," which reflects what we can infer from the results. That is, if these teachers do not use the potentials, then it is likely that the average teacher does not either.

4.3. DBR – empirical development of the guidelines

As explained in the Methods section, a didactic sequence was designed on the basis of a priori guidelines in Højsted (2020a) and then tested and redesigned in three design cycles. The design contained elements referring to each level of the guidelines; however, in the retrospective analysis, certain aspects were chosen and brought to focus for a fine-grained analysis in order to qualify the research outcome and keep the workload manageable. This resulted in three papers, zooming in on different aspects of the guidelines.

4.3.1. Zooming in on level 1 of the guidelines – dependency tasks

Paper III: *Analysing Signs Emerging from Students' Work on a Designed Dependency Task in Dynamic Geometry* (Højsted & Mariotti, 2020a).

The paper presents a fine-grained account of the task design principles of the initial tasks of the didactic sequence. The principles are presented in the structure of objective, hypothesis and choices, which we developed to ensure coherency and alignment between overarching learning objectives hypotheses about the types of tasks and choices made at the micro level of design. This structure also offers a systematic and explicit support for connecting the design process to the revisions suggested based on the empirical data.

The results elicit students' difficulties in predicting and interpreting the hierarchical nature of dependencies in DGE and show that the students, in fact, expect dependencies to be nonhierarchical. The data shows that high-achieving students describing the behavior of constructions during dragging refer to the specific elements of the figure (e.g., points or line segments) to describe the dependencies between geometrical properties of the figure, while low-achieving students refer to their global appearance, with their justifications referring neither to the construction process nor to geometrical properties. The suggested revisions convey that it is necessary to include specific prompts in the task design in order to shift the students' attention onto specific elements of constructions and that it seems necessary to explicitly ask students to explain any unexpected observation for active reflection to occur. While the role of the teacher is not in focus in the paper, some analyses and reflections are oriented toward the inappropriateness of the teacher in not facilitating an awareness of the impossibility of dragging the derived objects in the task.

4.3.2. Zooming in on students' proofs and the toolbox puzzle design

Paper IV: *A "Toolbox Puzzle" Approach to Bridge the Gap between Conjectures and Proof in Dynamic Geometry* (Højsted, 2020c).

Results from this paper concern the design of tasks from the latter stage of the didactic sequence that focus on supporting students' production of inferential arguments to verify conjectures that they have made in DGEs. The generic structure of the "toolbox puzzle" tasks is prediction and guided exploration leading to students' production of conjectures, followed by a proof as an explanation activity, using the toolbox. The toolbox contains axioms and theorems, as well as a support drawing, and serves the purpose of supporting the students in developing an inferential argument since they have no previous experience with this. Hence, looking for the explanation becomes solving the puzzle, using the pieces that are in the toolbox. The idea is then that after several such tasks, the toolbox can be empty.

The presented case of two high-achieving students indicates that the task design can foster an interplay between the production of conjectures in DGEs and deductive reasoning. The students argued theoretically to explain what they initially guessed and investigated empirically in the DGE. Importantly, the activity of theoretical validation seemed to make sense to them, even if they were already convinced by the empirical investigation. The students seemingly had to get acquainted with the structure of the toolbox puzzle approach before it became meaningful for them. Most students found it difficult to write down what they otherwise could explain verbally and by using gestures.

Medium- to low-achieving students struggled with deductive reasoning, with some only managing the conjecturing activity.

4.3.3. Zooming in on the role of the teacher

Paper V: *Guidelines for the Teacher – Are They Possible?* (Højsted & Mariotti, 2020b).

This paper reports on the design and implementation of teacher guidelines to manage classroom discussions in the last iteration of the DBR cycles. The guidelines were anchored in the TSM frame utilizing the notions of the didactic cycle and the four categories of teacher actions in an adapted version. More precisely, a condensed text was offered to the teacher introducing the didactic cycle and the four categories of teacher actions, followed by a table to be used in the classroom discussion in relation to each task. The table contained a description of (1) the intended mathematical meanings, which is the educational aim of each task; (2) the students' personal meanings expected to emerge from the activity; and (3) corresponding teacher action advice to the teacher with specific examples and comments. Results of the analysis coming from the first classroom discussion, as well as from an interview with the teacher, are presented. The results show that the teacher neglected two of the teacher actions and followed the guidelines only to a limited extent. A dilemma appears in the results – the teacher indicates that the guidelines must be shorter, while the data suggests that the teacher needs to grasp more of the theoretical frame or understand it better. It seems difficult to communicate theoretical assumptions using guidelines in this form, which indicates that the guidelines must somehow be flexible in relation to different teachers' pedagogical paradigms. A critical issue emerges from the results – how to communicate a theoretical frame in the form of condensed guidelines in light of teachers' different pedagogical paradigms?

4.4. Summary of results

The results from the papers may be coarsely summarized as follows:

Paper I

- Four DGE potentials in relation to reasoning competency were identified: dragging, feedback, measuring and tracing.
- The utilization of the potentials was described in three dimensions of a priori guidelines – students' cognition, task design, and the role of the teacher – and related to the expected

mobilization of reasoning competency. The guidelines convey a hierarchical learning trajectory.

Paper II

- There are indications that the potentials are scarcely used in Danish lower secondary schools and that DGE is used as a substitute of the paper and pencil environment.
- Teachers most often highlight the pragmatic value of techniques.
- Awareness of free and locked objects seems not to be a particular focal point, which is therefore addressed in the design of the didactic sequence.

Paper III

- Design principles are presented in the systematic structure of objective, hypothesis and choices.
- Danish lower secondary school students intuitively expect dependencies in DGEs to be nonhierarchical.
- High-achieving students refer to geometrical properties and specific elements of the constructions to justify their behavior during dragging, while low-achieving students refer to their global appearance and not geometrical properties.
- It seems necessary to include prompts that shift students' attention to specific elements of constructions and require explanations for unexpected observations in order for active reflection to occur.

Paper IV

- The "toolbox puzzle" task design can foster an interplay between the production of conjectures in DGEs and deductive reasoning.
- Theoretical validation seemed to be meaningful for the students in this approach.
- Students found it difficult to produce written arguments.
- Medium- to low-achieving students struggled with deductive reasoning, with some only managing the conjecturing activity.

Paper V

- It is difficult to communicate theoretical aspects in the form of guidelines, at least in the chosen design form of a condensed text and a table.

- Dilemma: The data suggest that more of the theoretical frame must be shared with the teacher and the importance of key aspects must be elaborated. However, according to the teacher, the guidelines must be shorter.
- The guidelines must somehow be flexible enough to be adapted to different teachers' pedagogical paradigms.

5. Discussion

This chapter examines pertinent issues, starting with the development of guidelines based on the theoretical and empirical results and providing a synthesis in the form of a model. Afterward, the guidelines are discussed, including how they were made and who might be possible stakeholders. The connection between the papers and their contribution toward the overall project, as well as certain findings from the papers, are discussed. Finally, I reflect on the suitability of the methodological and theoretical choices made, referring to notions of validity, reliability, generalizability and trustworthiness.

5.1. The development of guidelines – providing a synthesis

What exactly do the guidelines comprise, how did they come to be and how have they evolved?

The a priori guidelines comprised a theoretically developed hierarchical learning trajectory elaborating on students' cognition, task design and the role of the teacher in relation to the development of mathematical reasoning competency in DGE activities. The guidelines were developed based on previous research in the field and contained five steps of progression (0–4) that are unfolded in Højsted (2020a) and in the appendix attached to that paper. The steps related to students' cognition can briefly be summarized as follows:

(0) Having basic DGE proficiency – awareness of tools for construction and measurement.

(1) Discerning free and locked objects.

(2) Discerning direct and indirect invariants – awareness of certain dragging and measuring modalities for exploration and conjecturing.

(3) Exploiting DGE feedback to find counterexamples to conjectures and investigate pseudo-objects.

(4) Theoretically verifying the conjectures.

To each of the steps, there is an elaboration of types of tasks that may foster such progression, the role of the teacher in facilitating this evolution and which part of reasoning competency the activity is expected to mobilize.

These a priori guidelines that were theoretically designed initially were subsequently empirically revised based on findings from the DBR iterations. The refinement, which is spread out in the papers, is now condensed in the following. After having engaged in a fine-grained analysis of the empirical data related to level 1 in the guidelines (Højsted & Mariotti, 2020a), the toolbox puzzle approach at level 4 (Højsted, 2020c) and the role of the teacher (Højsted & Mariotti, 2020b), a succinct summary is made based on the analysis gained from the papers. Refinements are stated in relation to the three aspects that were chosen as analytical foci, rather than reiterating the whole learning trajectory from the first paper (Højsted, 2020a), and a condensed model is suggested.

Level 1 of the guidelines is reformulated into “developing an interpretative frame” that comprises being able to discern free and locked objects and possessing awareness of the hierarchical nature of dependencies in a DGE. Essentially, it concerns being able to interpret what happens on screen during dragging by means of mathematical reasoning referring to logical dependencies between geometrical properties as well as software design reasons. As seen in paper III, this ability is nontrivial, and students’ awareness of this fact can certainly not be taken for granted. It is, therefore, a necessary focus point to consider if the educational aim is to exploit the dragging tool. The data suggest that working with dependency tasks and construction tasks (Højsted, 2020a; Højsted & Mariotti, 2020a; Mariotti, 2012), using the design heuristic Predict–Observe–Explain (White & Gunstone, 2014), may, indeed, trigger an intellectual curiosity related to the functioning of the program and the geometry that is embedded. However, it seems necessary to explicitly ask the students to explain any unexpected observation for active reflection to occur or else they may just move on. It is essential in a classroom setting that the teacher is aware of the semiotic potential of DGEs and his mediating role in fostering the evolution of this interpretative frame. The designed task type “dependency tasks” is added to the task column.

The toolbox puzzle approach showed potential in connecting the development of DGE-based conjectures with that of deductive argumentation. The fact that the toolbox puzzle activity became a sensemaking activity for the students is promising and supports the approach of introducing proof as an explanation, at least in the context of students being required to undertake theoretical validation of DGE-generated conjectures that they are already empirically convinced of. Level 4 is reformulated in the task column so that it now incorporates “toolbox puzzle tasks.” Further refinements of the design may consider steps to support medium- to low-achieving students in developing inferential arguments and, generally, support students’ production of written argument.

The section concerning the role of the teacher is elaborated to include the structure of the guidelines produced for the teacher to manage the classroom discussion in the third iteration. The guidelines are developed to support the teacher in performing their mediating role. The guidelines comprise a condensed text introducing the didactic cycle and the four categories of teacher actions, as well as a table. The table describes the intended mathematical meanings, which the DGE tasks are meant to evoke; the expected student signs and underlying meanings, which are expected to emerge in the classroom discussion (based on previous experience and hypothetical scenarios); and the possible actions of the teacher in relation to these signs. As seen in paper V, the implementation of the guidelines was not particularly effective. In fact, the data suggest that it is difficult to communicate underpinnings of an elaborate theoretical frame such as TSM using the design of condensed guidelines (Højsted & Mariotti, 2020b). Nevertheless, the idea and structure of these teacher guidelines for managing classroom discussions still seems worth pursuing, although it calls for more reflection on how to interface with teachers so that the guidelines become effective. This agenda, which is at the core of the articulation between theory and practice, will be pursued further in forthcoming research. Meanwhile, the structure of intended meanings, expected meanings and teacher actions is incorporated into the guidelines.

Figure 6 provides a revised synthesis and a considerably condensed model of the guidelines for the design of didactic sequences that was initially presented in paper I.

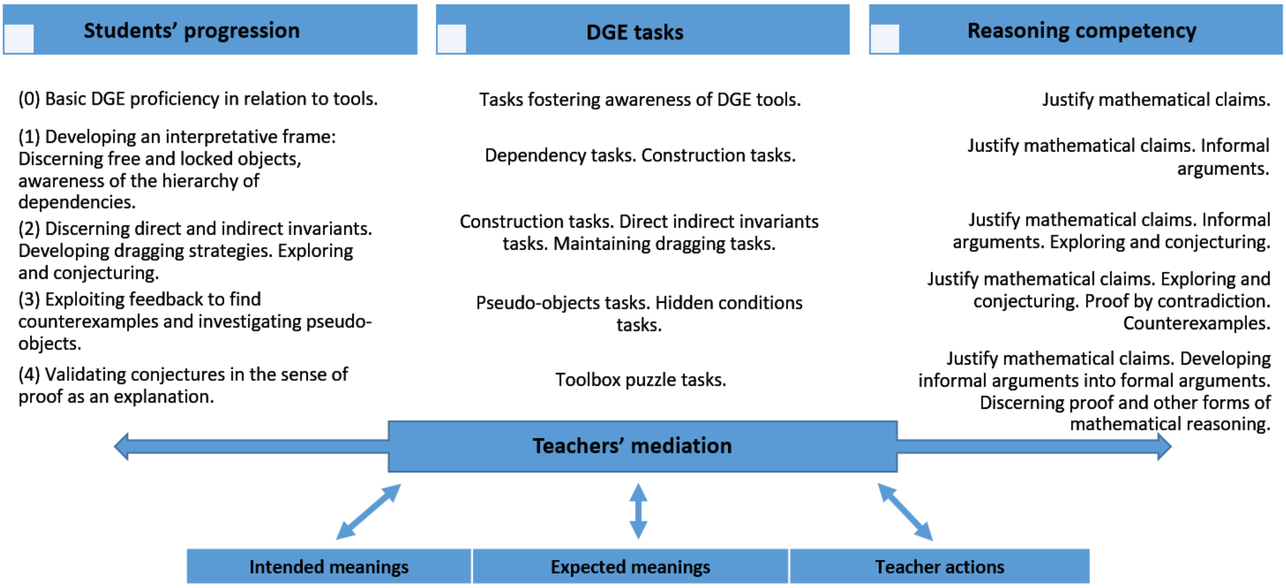

Figure 6. A condensed model of a posteriori guidelines for the design of didactic sequences to utilize the semiotic potential of dynamic geometry environments in order to foster students' development of reasoning competency (tasks described in Baccaglini-Frank et al., 2013, 2017, 2018; Baccaglini-Frank & Mariotti, 2010; Højsted, 2020a, 2020c; Højsted & Mariotti, 2020a, 2020b; Komatsu & Jones, 2018; Mariotti, 2012)

As was the case in paper I, the first column in the model indicates steps of progression going from 0–4; however, even if the steps are hierarchical, they are not to be considered as discrete or clearly continuous. For example, the development of several steps may occur at the same time, and they also overlap to some extent.

The second column describes the types of DGE tasks that can support each level of progression, while the third column indicates which characteristics of mathematical reasoning competency this work may mobilize.

The important mediating role of the teacher is modeled as going across the specific artefact tasks to the students' progression and toward reasoning competency. This symbolizes the role of the teacher in fostering the evolution of students' signs and underlying meanings that emerge in the DGE task work toward mathematical signs and meanings – which, in this case, are the constituents of reasoning competency. The mediation of the teacher is supported by the teacher guidelines, which comprise a description of the intended mathematical meanings that the task is meant to foster; the expected student signs and underlying meanings, which are hypothesized a priori and empirically developed;

and finally, the possible actions of the teacher, which are based on the four teacher actions presented by Bartolini-Bussi and Mariotti (2008).

The guidelines are formulated at a level above the concrete design of a didactic sequence, meaning that the development of a didactic sequence based on these guidelines requires that further choices are made in relation to the concrete sequence design. In that sense, the guidelines are **prior to a concrete sequence** and serve as the foundation or **guiding principles** for designing didactic sequences with DGEs when reasoning competency constitutes the educational aim. Further choices made in the design of the concrete didactic sequence are elaborated and exemplified in Højsted and Mariotti (2020a) and in section 3.1.3. of this dissertation. On the one hand, certain details that proved valuable in the data from the project are omitted in such a condensed model presented in Figure 6, which does not report on detailed design choices – for example, design choices such as working in pairs with requests of predicting and justifying assumptions to each other to foster reasoning competency or the utility of using the heuristic of Predict–Observe–Explain (White & Gunstone, 2014), while requesting explanations for any unexpected observation, to trigger active reflection. On the other hand, the synthesized guiding principles in the condensed model afford flexibility in relation to the concrete design and possible pedagogical approaches.

These revised guidelines constitute the outcome that addresses the project's answer to the third and final research question.

5.2. How is it possible to change praxis and is it reasonable to do so?

Who are the guidelines for? In principle, the guidelines may be useful for anyone interested in designing didactic sequences that utilize potentials of DGEs in relation to fostering students' development of mathematical reasoning competency. However, closing in on some imaginable stakeholders, then the current research format formulation of the guidelines may, in particular, be interesting for researchers in mathematics education and preservice mathematics education educators, perhaps also curriculum stakeholders or mathematics textbook authors. Adaptation and reformulation are likely necessary to reach lower secondary school teachers themselves. This is planned in further dissemination, which is explained in Chapter 6.

The important issue emerging from the results reported in paper V on the implementation of teacher guidelines for classroom discussions (Højsted & Mariotti, 2020b) deserves more reflection and further work to more effectively implement the structure of intended meanings, expected meanings and teacher actions. It seems more steps must be taken to elaborate on the TSM frame if these guidelines should become more effective. However, the problem of conveying an elaborate theoretical frame in a condensed manner seems difficult to solve, and the dilemma is that while more elaboration seems necessary, the teacher wants shorter guidelines. A possible path going forward could be to make the guidelines flexible, taking into account teachers' possible different pedagogical paradigms. Exactly what that would mean is not yet clear. Perhaps it may comprise identifying what the absolute core concepts that must be elaborated are and clearly conveying to the teacher that an awareness of these concepts is fundamental for the implementation of this type of didactic sequence. Another path could be to seek alternative ways of interfacing with teachers by, for example, other mediums than written text. The most obvious being the use of videos or some online interaction, perhaps in the form of video packages conveying the core pedagogical approach and communicating theoretical aspects with concrete examples from teachers acting in a classroom discussion. The role of the teacher has only been one of the focus points in this dissertation, and not too much data has been disseminated on this issue, perhaps leaving more questions unanswered than answered; however, the crucial role of the teacher for making the teaching material successful was clearly recognized from the collected data in the five classes. Supporting all teachers in successful implementation seems to be a critical issue.

Another point of discussion emerging from the results is whether it is reasonable to spend the required time in school mathematics education for students to become proficient in a DGE such as GeoGebra in order to foster the development of reasoning competency in relation to geometry, which may well be fostered in other ways (e.g., using a ruler and a compass). According to the guidelines, the development of an interpretative frame related to the DGE is important to perform some of the activities that are subsequently suggested – for example, searching for indirect invariants (cf. Lopez-Real & Leung, 2006). This interpretative frame comprises an awareness of the program rules that govern the DGE, which includes those that are a consequence of Euclidian theory, as well as rules that are choices made by the software developers and perhaps not that relevant from a mathematical point of view. From the midpoint task in the project data (Højsted & Mariotti, 2020a, 2020b), we can see that it is not a trivial task and that students intuitively expect dependencies in DGEs to be nonhierarchical. The necessity of an interpretative frame is also indicated by Duval (2017):

> the instrumental construction of shape figures, particularly using software, impart a reliability and objectivity to them allowing to use them for heuristic experiments. But here too, for that

> kind of activity, "seeing" is important because the effective use of a tool requires that we anticipate what can be done and obtained. (p. 58)

The following questions then arise: How much time and effort is reasonable to spend on learning the intricacies of a software in order to reach the level necessary to be able to perform activities that actually comprise the mathematical goals? And is it reasonable to expect that teachers will possess this interpretative frame, knowing the difference between what is mathematically relevant and what is not? Of course, this deliberation is relevant for the use of any digital or nondigital tool in mathematics education, including the ruler and compass. Compared to many other tools, GeoGebra is, in fact, rather intuitive and is an easy tool, according to the teachers (Højsted, 2020b).

Another way of viewing this discussion point is that mathematics, as a discipline, has developed in tune with technology and the natural sciences through history (Misfeldt, 2013), as have the means by which we gain access to mathematical objects. It could be argued that dragging in DGEs is indeed a mathematical activity and being able to justify the behavior of objects during dragging in DGEs, including the behavior that is not a consequence of Euclidean theory, can be considered as a mathematical activity. If this premise is accepted, then students (and teachers) must learn the rules of the "new game" for experiencing geometry (cf. Leung, 2015; Lopez-Real & Leung, 2006).

5.3. Revisiting the conceptual frameworks

It is engaging to consider, in retrospect, what the affordances and limitations of the DBR-related choices that were made are – in particular, the connection between DBR and the conceptual frameworks that were utilized (KOM's mathematical reasoning competency and TSM).

The choice of describing the educational goal using mathematical reasoning competency from the KOM framework required due consideration during the initial phase of the project – in particular, during the review process and the development of a priori guidelines. Mathematical reasoning competency does not address a particular topic, which may be more common in DBR, and reasoning competency is not commonly used in DGE research internationally; therefore, there was a need for operationalizing the competency, compartmentalizing it into a form that could be useful to identify pertinent previous research and develop a priori guidelines. The reasoning competency approach thoroughly influenced the development of guidelines and the design of the didactic sequence, not only in relation to the types of tasks that were deemed suitable but also concerning the choices made at the level of task formulation, such as requesting the students to justify claims to each other. The

Predict–Observe–Explain design heuristic proved valuable in relation to reasoning competency since it inherently requires justifications from the students. In summary, although the use of the KOM's mathematical reasoning competency required operationalization and compartmentalization to be used in the DBR approach, there were no conflicts or difficulties as such.

To what extent is the TSM frame and DBR compatible? Reflecting on this issue, it seems that the choice of anchoring the project in the TSM frame gave occasion to strokes of serendipity in relation to the DBR part of the project because there were overlaps in certain aspects. TSM and DBR both contain elements of didactic design, and the design is grounded in an a priori analysis. Hence, the hypothetical learning trajectory, from DBR methodology, coincides with the analysis of the semiotic potential of the artefact (DGE in this case) from the TSM frame. The same overlap is applicable in the development of tasks designed to exploit the semiotic potential of DGEs uncovered in the a priori analysis. The proposed analysis of the unfolding of the semiotic potential from the TSM frame also coincides with the retrospective analysis from DBR.

Another beneficial output from anchoring the DBR in TSM in relation to DBR lies in the analytical focus of TSM on the emergence of signs and the appreciation of all relevant forms of signs (gestures, verbal utterances, written products or on-screen DGE actions) produced by the students. This corresponds well with the use of data triangulation, which is advised in DBR (see section 5.4.1. on internal validity).

The TSM frame also proved valuable in determining which aspects were to be developed in the DBR – particularly the focus in the TSM frame on the role of the teacher, which influenced the design of the teacher guidelines.

5.4. Revisiting methodological choices

As presented in Chapter 3, the methods applied in the project are threefold, used in a multiphase design (Creswell & Clark, 2011) with a qualitative priority, and the results play different roles in the pursuit of the overarching research goals. A reasonable question that arises is why this methodological choice? The justification is that this approach was deemed suitable considering the research questions: (1) To bring to light potentialities of DGE in relation to reasoning competency; no other approach other than a literature review seems relevant. (2) To uncover the extent to which these potentials are utilized in lower secondary schools in Denmark requires data from across the country, and a survey can provide exactly that. Additional qualitative studies could have further qualified the answer to the second research question by, for example, conducting follow-up interviews with teachers from the survey. It would, in fact, have been interesting to interview one teacher from each of the categories in paper II (Højsted, 2020b). However, it was deemed to require too many resources, considering other important aspects of the project. (3) DBR was considered suitable in the quest to develop guidelines for teaching that utilizes potentials of DGEs in relation to supporting students' development of reasoning competency because it serves "an explanatory and advisory aim – namely, to give theoretical insights into how particular ways of teaching and learning can be promoted" (Bakker & van Eerde, 2015, p. 431). Since teaching that utilizes potentials of DGEs in relation to reasoning competency in lower secondary schools was not expected to be found in the current praxis, it was necessary to design such activity in order to be able to study it.

The aligned quantitative and qualitative studies build on what was learned previously to address the overarching research objective. The papers, which are a consequence of the methodological choices, are, therefore, linked in several ways.

5.4.1 The interconnectedness of the papers

The relationship between each individual paper and the overall project may be highlighted referring to the research questions. The first paper addresses research question 1, the second paper addresses research question 2 and papers III, IV and V all address research question 3.

However, the papers are also interconnected. The first paper is clearly connected to the DBR part of the project because the first paper entails not only review results but also an unfolding of the a priori guidelines that are utilized in the DBR part of the project. Hence, paper I can be conceived of as the preparatory phase that paves the way for papers III, IV and V. Paper I is, in fact, also connected to paper II because the potentials that were unveiled in the review laid the foundation of the development of the survey, which is the research focus in paper II. Since paper II examined the practice of lower secondary school teachers with regards to DGE potentials, it provided insights that were taken into consideration in the DBR part of the project. In fact, the task design that is in focus in paper III on dependency relationships in a DGE is a direct consequence of the survey results in paper II, which indicated that dependency relationships mediated by DGE were not a point of focus in Danish lower secondary schools. Another methodological choice concerns the mathematical aim of the project.

5.4.2 Using the whole of mathematical reasoning competency as the learning aim

As described in Chapter 2, mathematical reasoning competency entails several characteristics and can be considered a quite broad learning aim. The choice of focusing on all characteristics of reasoning competency, therefore, led to a learning trajectory that contained many layers of learning aims. One could, in fact, argue that the learning aim was too broad because thoroughly addressing each level of the guidelines would require the development of a didactic sequence that would stretch over a long-term experiment rather than the 15-lesson intervention that was developed in the project. Even the 15-lesson sequence that was tested and redesigned in 5 different classes produced so much data that I decided to analytically focus my attention on certain parts – that is, zooming in on level 1 of the guidelines, the toolbox puzzle approach and the role of the teacher in classroom discussions. Hence, on the one hand, the choice to maintain mathematical reasoning competency in its entirety as the learning goal resulted in a learning trajectory that was broad in relation to the allocated resources in the project, and as a consequence, I had too much data. In retrospect, it may well have been sufficient to focus my attention only on some parts of reasoning competency in the project – for example, on the ability to transform heuristic reasoning into actual proofs. It may also have been sufficient to focus only on task design or perhaps only on the role of the teacher. However, on the other hand, the guidelines provide a holistic coherent learning trajectory of DGE utilization aiming to foster students' development of mathematical reasoning competency, including important aspects of the teaching and learning process (i.e., students' cognition, task design and the role of the teacher). As a research outcome, this seems to be a quite useful contribution to the research field as well as for stakeholders (such as preservice mathematics teacher educators, textbook developers and mathematics educators at other levels) – even if the guidelines are primarily theoretically anchored and only, to some extent, empirically refined. Further empirical work and analysis of the data are already ongoing. Taking these aspects into consideration, I am content with the chosen approach.

5.4.3. Validity and reliability in DBR

Bakker and van Eerde (2015) propose that validity and reliability can be addressed in DBR studies using the notions of internal and external validity, as well as internal and external reliability.

Internal validity "refers to the quality of the data and the soundness of the reasoning that has led to the conclusions" (Bakker & van Eerde, 2015, p. 444). A technique that may be applied to improve the internal validity of a DBR study is the use of data triangulation in the retrospective analysis (Bakker & van Eerde, 2015). The many sources of data (transcripts, video, screencast, written products and teacher interviews) collected in the project allowed for this type of analysis. In fact, the theoretical anchoring in the TSM, which encourages an analytical focus on the emergence of signs in all relevant forms of expressions (e.g., gestures, verbal utterances, written products or on-screen DGE actions), facilitates that data triangulation is applied. In terms of internal validity, this points to an inherent analytical strength of the TSM frame.

External validity refers to the generalizability and transferability of the results. These issues concern questions about

> how we can generalize the results from these specific contexts to be useful for other contexts […] The challenge is to present the results (instruction theory, HLT [hypothetical learning trajectory], educational activities) in such a way that others can adjust them to their local contingencies. (Bakker & van Eerde, 2015, p. 444)

The guidelines can be considered a Danish context-specific theoretical contribution concerning the utilization of DGE potentials in relation to fostering students' development of reasoning competency empirically refined in certain aspects. However, as mentioned in Chapter 1, the notion of competencies is gaining traction in curricula around the world, even if different terminology may be used. Therefore, the guidelines may easily be adjusted to contexts situated in countries that have adopted learning aims that resemble reasoning competency or contain elements of reasoning competency as well as learning aims concerning DGE and geometry.

Some of the results reported in the papers are of a general nature and can reasonably be conceived to be nonspecific to the Danish context (e.g., from paper III – the students' intuitive expectations concerning dependencies in DGE, low-achieving students' reference to global appearance of figures and the promising potential of the design heuristic Predict–Observe–Explain in relation to reasoning competency).

Internal reliability in DBR "refers to the degree of how independently of the researcher the data are collected and analyzed" (Bakker & van Eerde, 2015, p. 445). Bakker and van Eerde (2015) mention two techniques to improve internal reliability: to collect data with objective devices and discuss episodes of data with colleagues for peer examination. In the project, these techniques were utilized, on the one hand, by collecting data with video, screencast and audio devices and, on the other hand, by extensively discussing episodes of data with the coauthor of papers III and V.

External reliability concerns the replicability of the study – which, in the context of DBR, requires "that it is clear how the research has been carried out and how conclusions have been drawn from the data" (Bakker & van Eerde, 2015, p. 445). In the Methods section and the dissemination of the results of the research project, attempts were made to report on the process as thoroughly and candidly as possible (to the best effort of the author) so that the reader, in a transparent manner, may follow the project's process and results.

5.4.4. Trustworthiness

Another conducive perspective to reflect on the methodological quality of the study is Schoenfeld's (2007) concept of *trustworthiness*. Generally, when assessing the quality of a study, the concepts of validity and reliability are used (Blaxter et al., 2010). These notions are used in the previous section; however, the notion of trustworthiness is also addressed here, complementing the prior elaboration.

Trustworthiness includes five subconcepts – the first two of which set requirements for mathematical didactical conceptual frameworks, while the last three, are more relevant for assessing the quality of empirical research. The first subconcept is what Schoenfeld (2007) calls *rigor and specificity*, and it contains the idea that one must be consistent, precise and thorough when describing theoretical concepts or actions that are to be used in relation to a study. Considering the rigor and specificity of the project, I have unfolded the two conceptual frameworks (KOM's reasoning competency and the TSM) in Chapter 2, describing the constituents that are relevant for the project in relation to the design of each study and in terms of analyzing the data in the papers of the dissertation. In the Methods section, I have described how the design of the different studies was based on the understanding and operationalization of the concepts from the KOM framework and the TSM.

The next subconcept, *replicability* (Schoenfeld, 2007), concerns the consideration that a study must be presented in enough details so that key aspects of a study could be repeated or further developed by others. As mentioned before, to adhere to this concept, I tried to present the methodology comprehensibly by including details that may seem not too important yet contributing to make it clear to how each study was performed. The details permit that others, will understand justifications and, if they prefer, build on the ideas, which is precisely the idea in the requirement of replicability.

The subconcept *multiple sources of evidence* (Schoenfeld, 2007) is about triangulation. It considers the fact that it is better for a study to have multiple perspectives on the same object of study because something that is visible through one lens is not always visible through another. As mentioned in the previous section, there were many sources of data in the DBR part of the project (transcripts, video, screencast, written products and teacher interviews), facilitating that data triangulation was applied.

The same cannot be said in relation to the questionnaire; however, the inclusion of open-ended questions did provide qualitative nuances that could complement the quantitative data. The last subconcept is related to the *generality and importance* (Schoenfeld, 2007) of a study, which concerns how widely the results are applicable as well as whether the study is important.

We can discuss the generalizability of the results gathered from the survey (Højsted, 2020b). As mentioned in the Methods section and in paper II, the population of the survey was not representative of the average Danish lower secondary school mathematics teacher. In fact, the respondents may be considered as “super users” because of the way the survey was distributed and because their reporting on GeoGebra usage was high. The case resembles what Flyvbjerg (2006, p. 14) denotes as a *critical case* – which “can be defined as having strategic importance in relation to the general problem” because it can provide the possibility to generalize in the form of “if it is valid for this case, it is valid for all (or many) cases” and, conversely, “if it is not valid for this case, then it is not valid for any (or only few) cases” (Flyvbjerg, 2006, p. 14). Using this rationale, we can generalize by inferring that since the “super users” in the survey did not use the potentials to a great extent, then the average lower secondary school teacher likely does so to an even lesser extent. The results from the survey are important because while there are indications that GeoGebra is widely used, no previous research has analyzed *how* it is used in Danish lower secondary schools.

The results presented from the cases in papers III, IV and V related to DBR are not claimed to be generalizable, even though some results may well be. Rather, these results serve as *an existence proof* (Schoenfeld, 2007) of what is possible in average/above-average Danish lower secondary school classes with this approach and serve to confirm/invalidate the hypothesized learning trajectory. While the results from the cases may not be generalized as quantitative studies can, they can give valuable insights from one specific context that may be transferable to other contexts. In fact, good case narratives can provide irreducible quality and capture phenomena that cannot be captured by quantitative data (Flyvbjerg, 2006), such as the findings described in paper III (Højsted & Mariotti, 2020a) that students intuitively expect dependencies in DGEs to be nonhierarchical and high-achieving students refer to geometrical properties and specific elements of the constructions to justify their behavior during dragging, while low-achieving students refer to their global appearance and not geometrical properties. This is also exemplified in paper IV’s findings (Højsted, 2020c) that the “toolbox puzzle” task design can foster an interplay between the production of conjectures in DGEs and deductive reasoning while being a meaningful activity for students. Additionally, providing insights into the difficulty of sharing with a teacher the underpinnings of an elaborate theoretical frame in condensed teacher guidelines from paper V.

In conclusion, while some generalization can be made in relation to the survey, the DBR studies offer limited *warranted generality* in the form of *existence proofs* (Schoenfeld, 2007, p. 89). However, Schoenfeld (2007) asserts that such studies can “bring important issues to the attention of the field, make theoretical contributions, or have the potential to catalyze productive new lines of inquiry” (p. 89). I would argue that this is the case with the local theoretical contribution concerning DGEs in relation to the reasoning competency provided in this project – which, hopefully, can contribute toward better DGE utilization in Danish lower secondary schools and be useful for other contexts.

6. Conclusion

In this dissertation, I set out to answer the following three overarching research questions:

1. *What are the potentials of DGEs in relation to supporting students’ development of reasoning competency?*

2. *To what extent are the potentials currently utilized in Danish lower secondary school?*

3. ***Which research-based guidelines feature in the design of teaching that utilizes DGE potentials in order to support students’ development of reasoning competency?***

I have investigated these research questions, which were refined after adopting the theoretical frame of the TSM, by engaging in mixed methods research studies, using a multiphase design with a qualitative priority. I have reported on these studies in five individual papers – which are connected and, as a whole, have contributed to addressing the research questions in the form of theoretical and empirical contributions.

Addressing the first research question, I identified from the review that the potentials of feedback, dragging, measuring and tracing in DGEs may be exploited in relation to fostering students’ development of mathematical reasoning competency. The potentials are not separate domains; in fact, dragging may be considered a crucial potential that intervenes in the other three potentials.

To answer the second research question, I found indications from the survey that these potentials are scarcely utilized in Danish lower secondary schools, that DGEs are used to solve paper and pencil tasks and teachers highlight the pragmatic values of techniques afforded by the software.

Finally, in relation to the third research question, from the DBR approach, I identified guidelines in three dimensions describing a learning trajectory in terms of students' cognition, types of DGE tasks that utilize the potentials and mobilize different characteristics of reasoning competency and an initial structure of guidelines for the teacher to manage classroom discussions. The guidelines were initially developed theoretically and, in part, refined empirically.

The guidelines were formulated at a level that can serve as the guiding principles for the design of didactic sequences to utilize potentials of DGEs in order to foster students' development of reasoning competency. Hence, certain choices must be made in the actual utilization of the guidelines, which means there is flexibility in relation to the concrete design. The choices and rationales that were made in the project design are reported in part in Højsted and Mariotti (2020a) and in part in section 3.1.3. – one choice being the use of the design heuristic of Predict–Observe–Explain (White & Gunstone, 2014), which proved promising.

7. Contribution to the research field and praxis

Without reiterating all the results, this project's novel research contribution can coarsely be summarized into two main constituents: First, the project provides insights into the actual DGE usage of Danish lower secondary school teachers, showing that the potentials in relation to reasoning competency are scarcely utilized and teachers mainly refer to the pragmatic value of techniques performed with DGEs. Although Vejbæk's (2011) report on data from 2009 investigated *if* DGEs were being used, no up-to-date quantitative research exists on *how* teachers actually incorporate DGEs in Danish lower secondary schools. Internationally, the findings are in alignment with results in Bozkurt and Ruthven's (2017) and Ruthven et al.'s (2004) qualitative studies, which showed that teachers mainly refer to the added pace and productivity that technology provided.

Second, the project provides a Danish context-specific theoretical contribution (which is empirically refined at certain levels) in the form of guidelines containing a holistic learning trajectory in relation to the design of didactic sequences that utilize potentials of DGEs for the purpose of fostering lower secondary school students' development of mathematical reasoning competency. While there are studies that have developed task design principles or models in relation to DGEs (e.g., Fahlgren & Brunström, 2014; Lin et al., 2012; Komatsu & Jones, 2018; Olsson, 2019), none have done so in the particular context of fostering the development of reasoning competency.

Thus, the project provides some insights in relation to Niss et al.'s (2016) call for the need to understand the particular factors and contexts that help foster the development of mathematical competencies.

The next step in disseminating the project results is concentrating on reaching praxis. As a stroke of fortune, a new national center for the development of mathematics teaching, abbreviated NCUM (www.ncum.dk), was recently established in Denmark. The purpose of the center is to disseminate knowledge about mathematics teaching, making applicable research-based knowledge available to teachers and educators. The center plans to do so, in part, by developing "knowledge packets" that teachers can use. The results from this project can suitably be adapted and fed into one such knowledge packet that may advice teachers on how to utilize potentials of DGEs in relation to fostering students' development of reasoning competency, thereby contributing to the presence of thoughtfully designed DGE teaching/learning sequences and hopefully creating "marvels" in Danish lower secondary school mathematics.

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Paper I

Guidelines for utilizing affordances of dynamic geometry environments to support development of reasoning competency

INGI HEINESEN HØJSTED

This article reports on guidelines developed based on an extensive research literature review investigating the potentials of dynamic geometry environments (DGEs) when the educational aim is to support students' development of mathematical reasoning competency. Four types of potentials were identified – feedback, dragging, measuring, and tracing – and used in three dimensions of guidelines: students' cognition, task design, and the role of the teacher. Using constructs from the *Instrumental approach*, the *Theory of semiotic mediation*, and the van Hiele model of levels, affordances and guidelines are elaborated upon and their potentials for reasoning competency are analyzed.

Research on dynamic geometry environments (DGE) affordances has revealed potentials regarding the development of students' mathematical reasoning (e.g. Leung, 2015; Edwards et al., 2014). This is promising, because there is research in Denmark and internationally indicating that students' reasoning abilities are inadequate (e.g. Jessen et al., 2015; Hoyles & Healy, 2007). ICT is accessible at all levels of the Danish educational system, so, in principle, the potentials are available in the mathematics classrooms. However, students' access to DGEs does not guarantee greater learning outcome. The manner in which DGEs are used is essential (Jones, 2005). Therefore, it is an important research objective to develop guidelines for fruitful teaching with DGEs.

Since DGEs can be used for different purposes, it is necessary to clarify the mathematical aim of the teaching guidelines. The notion of mathematical competencies, which has gained substantial traction in mathematics education, can be used for this purpose. Niss et al. (2016) call for research into teaching that can support students' development of mathematical competencies.

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> Fostering, developing and furthering mathematical competencies with students by way of teaching is a crucial [...] priority for the teaching and learning of mathematics [...] We now need to understand the specific nature of the contexts and other factors that help create such progress. (Niss et al., 2016, p. 630).

Focusing on reasoning competency in relation to using DGEs, this study aims to contribute to this end by asking: *Which research-based guidelines may be formulated for teaching with DGEs in order to support students' development of reasoning competency*?

The main question gives rise to two auxiliary questions: (i) *Which affordances of DGEs can be considered as potentials when the educational aim is to support students' development of reasoning competency*? In addition, (ii) *Which dimensions should such research-based guidelines entail*?

To address these questions, a review is undertaken of existing DGE research, using reasoning competency as the searching and sorting lens. The review findings serve a dual purpose – to establish which dimensions the guidelines should entail, and to identify DGE affordances that can be considered potentials for reasoning competency. Theoretical constructs from the literature that are found to be useful in the conceptual development of the guidelines are also included. On the basis of this work, an analysis of the possible development of reasoning competency in relation to DGEs is conducted, and finally, to answer the main research question, guidelines are suggested. Since reasoning competency plays a crucial role in the article, an elaboration of the notion is in order.

The KOM framework and its reasoning competency

The KOM framework[1] introduces a competency-based approach comprising eight mathematical competencies, illustrated in the so-called KOM flower (figure 1). The framework is integrated in the Danish mathematics education curriculum, and has also had an impact on mathematics education around the globe (e.g. OECD, 2017; for a detailed account, see Niss et al., 2016).

In the reasoning competency (hereinafter referred to as RC), reasoning is defined as *"a chain of argument [...] in writing or orally, in support of a claim"* (Niss & Højgaard, 2011, p. 60). RC consists of the ability to create and present formal and informal arguments, as well as the ability to follow and evaluate arguments made by others. It involves understanding what a mathematical proof is, the role of counterexamples, and the difference between a proof and other forms of mathematical reasoning, such as explanations based on examples. In addition, it includes the ability to develop an argument based on heuristics into a formal proof. RC

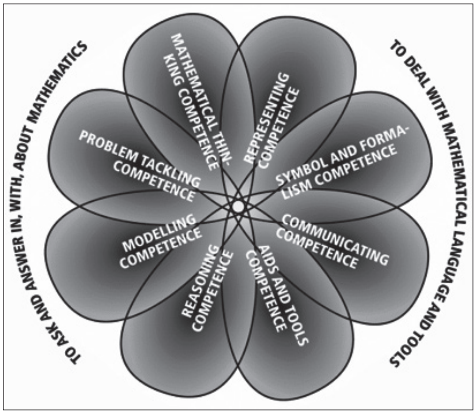

Figure 1. *KOM flower (Niss & Højgaard, 2011, p.1)*

is not only about justification of mathematical theorems, but also about creating and justifying mathematical claims in general, such as answers to questions and solutions to problems (Niss & Højgaard, 2011). In this article, the notion of proof is understood as the product of a proving process (Mariotti, 2012), which includes exploration and conjecturing as well as proving conjectures.

A person's attainment of a competency is qualified in three dimensions: *Degree of coverage* indicates to what extent the characteristics of the competency can be activated. *Radius of action* refers to the situations and contexts in which the competency can be mobilized, while *technical level* describes how advanced the mobilization is (Niss & Højgaard, 2011). To exemplify with regard to RC, a person might be able to follow reasoning put forward by others, but unable to put forward reasoning herself, thereby lacking in *degree of coverage*. She might be able to follow mathematical reasoning in the area of statistics but not in geometry and therefore has a limited *radius of action*. She might be able to follow complicated and technically advanced reasoning and therefore has high a *technical level*. The dimensions have a subjective character, since, for example, a high technical level depends on a person's age and peers.

Review method

The review was anchored in the hermeneutic framework for literature reviewing (Boell & Cecez-Kecmanovic, 2010, 2014), which fundamentally perceives the literature review as a non-linear process of gradually developing an understanding of and insights into a domain of research. The approach consists of two intertwined hermeneutic circles, the *search* *and acquisition* circle and the *analysis and interpretation* circle (see figure 2). The steps in the circles are carried out in an iterative process, thereby approximating a deeper understanding of the area of interest.

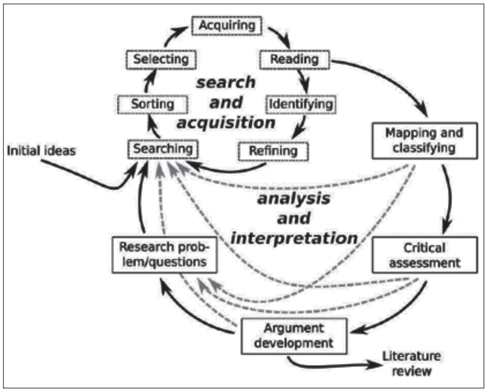

Figure 2. *The hermeneutic framework for the literature review process (Boell & Cecez-Kecmanovic, 2014, p.264)*

Applying the method

The i*nitial idea* was to operationalize RC into search words to cover central characteristics of the competency, and combine these with synonyms of dynamic geometry. The initial *search* was made in the MathEduc[2] and ERIC[3] databases, using the search words ”dynamic geometry”, ”geometry software”, ”geometry technology”, ”interactive geometry” and ”proof”, ”reasoning”, ”conjecture”, and ”justify”, which, after *sorting* and checking for items occurring twice, gave a total of 151 items. 62 items were found to be irrelevant after studying their abstracts, giving a total of 89 items that were *selected* and *acquired* to be read. Furthermore, proceedings from the CERME[4] technology TWGs were searched. After *reading* literature acquired from the primary search, interesting references were *identified* and followed (citation tracking) (Boell & Cecez-Kecmanovic, 2014) and, if suitable, added to the review. In addition, after reading and gaining some insight into the area of interest (*mapping and classifying*), adjusted search words were used in focused searches, and the operationalization of RC was *refined* with the search words ”counterexample”, ”argumentation”, and ”heuristic proof” in combination with synonyms of dynamic geometry. Other focused searches were related to theory, specifically ”instrumental genesis” and ”semiotic mediation” and ”Hiele” combined with dynamic geometry. A total of 136 publications were included to be examined in the review. The definition of RC played a decisive role in the review process, as it influenced the choice of search words and the *sorting* and *selection* of literature, and was the perspective used in the *critical assessment* of the mapped literature, helping to decide which DGE potentials and dimensions of guidelines were relevant. The *argument development* which is the synthesizing result of the literature review is unfolded in the following chapter.

Results

The aim of the review process was to address two issues (corresponding to the two auxiliary research questions): (i) to find the potentials of DGEs in relation to RC, and (ii) to inform the development of guidelines for teaching, i.e. use the literature to understand what dimensions the guidelines should entail, including which theoretical constructs may prove useful for this endeavor.

In the following section, theoretical constructs are introduced that were identified in the review to be useful in sharpening the guidelines conceptually. Then the argument development leading to the potentials is presented, followed by review findings leading to the dimensions of the guidelines, which are unfolded subsequently. Figure 3 provides an overview of the structural development of the guidelines that will be presented in the following sections.

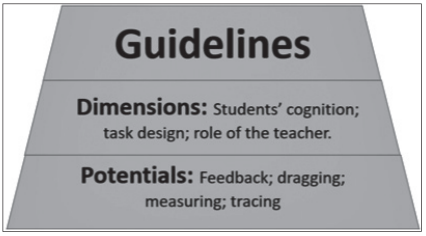

Figure 3. *Structural argument development of the guidelines*

Three theoretical findings to be used in the guidelines

Many studies from the review (e.g. Bretscher, 2009; Alqahtani & Powell, 2015; Gómez-Chacón, 2012; Gómez-Chacón et al., 2016) are embedded in the Instrumental Approach, which involves the process of instrumental genesis (Artigue, 2002; Guin & Trouche, 1999). According to this point of view, an artefact (such as DGEs) is not an instrument for a student from the outset, but becomes an instrument when the student can use the artefact in some meaningful way (Vérillon & Rabardel, 1995). Building on

Vergnaud's (2009) notion of schemes, instrumental genesis characterizes how subjects develop utilization schemes, which are cognitive schemes intertwining technical knowledge and mathematical knowledge. In a scheme-technique duality (Drijvers et al., 2013), these mental schemes evolve along with instrumented techniques for using the artefact to solve specific tasks. The instrumented techniques are the observable manifestation of the students' utilization schemes.

Several DGE studies (e.g. Falcade et al., 2007; Mariotti, 2012; Ng & Sinclair, 2015) are anchored in the *Theory of semiotic mediation* (Bartolini-Bussi & Mariotti, 2008). Emerging from a Vygotskian (1978/1934) perspective, the theory describes how teachers can exploit the possible ways of using an artefact (such as DGEs). In the semiotic perspective, the teaching and learning process is characterized as an evolution of signs such as gestures, verbal utterances, or DGE-mediated actions. The theory addresses students' initial production of *situated signs* as the artefact is used and on the following evolution into *mathematical signs*, which can be mediated by the teacher through social interaction as the teacher connects mathematical meaning to the evoked signs. Bartolini-Bussi and Mariotti (2008) use the notion of the semiotic potential of an artefact to describe the duality of emergent personal meanings and the possible mathematical meanings evoked by using an artefact. Mariotti (2012) considers the analysis of the semiotic potential of an artefact to be the core of any teaching design, and exploiting the potential involves:

> the orchestration of didactic situations where students face designed tasks that are expected to mobilise specific schemes of utilisation [...] the orchestration of social interactions during collective activities, where the teacher has a key role in fostering the semiotic process required to help personal meanings, which have emerged during the artefact-centred activities, develop into the mathematical meanings that constitute the teaching objectives" (Mariotti, 2012, p. 170)

Similarly to other DGE research (e.g. Jones, 2000; Idris, 2009; Kaur, 2015; Forsythe, 2015), utility is found in the van Hiele model of levels in order to relate the students' cognitive progression to a model of mathematical thinking. Van Hiele (1986) outlined how students' progress through five levels of mathematical thinking. The hierarchical structure has been criticized, as studies have found that students can be at several van Hiele levels in different situations (Burger & Shaugnessy, 1986). However, the levels may be thought of as different modes of thinking (Papademetri-Kachrimani, 2012; Forsythe, 2015) that can be activated in different situations. The latter understanding of the levels is adopted in this article and used in relation to students' cognition. Levels 1–4, which are relevant for this article, are elaborated upon: (1) *Recognition*. Students visually recognize figures intuitively by their global appearance. (2) *Analysis*. Students can describe the properties of a figure, but do not interrelate properties of figures. (3) O*rdering*. Students order the properties of figures by short chains of deductions and understand the interrelationships between figures. (4) *Deduction*. Students understand deduction and the role of axioms, theorems, and proof.

Potentials of DGEs in relation to RC

As a result of the review, four types of DGE affordances were identified as potentials[5] regarding students' development of RC: feedback, dragging, measuring, and tracing. DGEs are designed to mimic theoretical systems, such as Euclidean geometry, essentially creating a microworld in which activities follow the theoretical system governing the environment (Balacheff & Kaput, 1997). This signifies the existence of an inherent *feedback* function in the environment, since only objects which are possible in Euclidean geometry can be constructed. In a pencil and paper environment, there is no control on behalf of the paper over impossible constructions, allowing for imprecision, for example in a triangle where the medians do not intersect in the same point. Furthermore, dynamic geometrical figures can be constructed in the environment, so that certain properties are conserved when the figure is manipulated by use of the *drag* mode. The relationship between the elements of the figure is locked in a hierarchy of dependencies determining the outcome of a dragging action (Hölzl et al., 1994). This allows students to explore the figure by dragging free points to discover invariant properties of the figure, i.e. properties that are conserved. In a ”robust” construction, the properties are conserved when free points are dragged. On the contrary, in a ”soft” construction, not all properties are conserved (Healy, 2000; Laborde, 2005a).

Types of invariants have been classified to elaborate their role in conjecturing and reasoning (e.g. Leung, 2015; Baccaglini-Frank & Mariotti, 2010). Baccaglini-Frank and Mariotti (2010) suggest discernment between direct invariants, which are invariants in the construction that are defined directly by DGE commands used to complete the construction, and indirect invariants, which are those that arise as a consequence of the theory of Euclidean geometry, which governs the DGE. If a student is aware of the direct invariants of a construction and through exploration discovers indirect invariants, the activity might lead the student to make a conjecture (this will be discussed further in the section on task design). Many DGEs contain *measuring* tools that allow students to take measurements of, for example, angles, lengths, areas, and perimeters of constructions. If free points of the construction are dragged, causing the measures to change, the measurements are updated instantly and continuously. Therefore, it is possible for the students to discover invariant relationships between measures (Olivero & Robutti, 2007). In addition, many DGEs contain the possibility of *tracing* an object, so that the path can be visualized from a dragging action. In this way, tracing combined with dragging can be used to discover underlying invariant relationships (Baccaglini-Frank & Mariotti, 2010; Leung, Baccaglini-Frank & Mariotti, 2013). The affordance of visually representing geometric invariants when using the drag mode is considered a key feature of DGEs in relation to the development of mathematical reasoning, the ability to generalize results, and conjecturing in geometry (e.g. Arzarello et al., 2002; Laborde, 2001; Leung, 2015; Baccaglini & Mariotti, 2010; Edwards et al., 2014), which are some of the characteristics of RC.

Dimensions of the guidelines

The review showed that since its introduction, DGE research has had shifts in focus (see for example Jones, 2005; Mariotti, 2006; Laborde et al., 2006; Hollebrands et al., 2008; Olive et al., 2009; Sinclair & Robutti, 2013). In broad strokes, three dimensions of research could be identified. Initially, research focused on the learner, with some early contributions addressing student cognition (e.g. Arzarello et al., 2002; Hölzl et al., 1994). More recently, focus has shifted to design of adequate tasks to meet learning aims (e.g. Lin et al., 2012; Komatsu & Jones, 2018; Fahlgren & Brunström, 2014), as well as to the role of the teacher (e.g. Mariotti, 2006; Bartolini-Bussi & Mariotti, 2008). Sinclair et al. (2016) state that although research on DGE affordances is vast, task design and teacher practice remain understudied, a statement echoed by Komatsu and Jones (2018).

Findings from all three dimensions are relevant in relation to developing guidelines for teaching. Consequently, it was decided that the research-based guidelines should encompass findings regarding students' cognition, task design, and the role of the teacher.

Students' cognition

Several studies on students' cognition in DGE-related work are embedded in instrumental genesis (e.g. Leung et al., 2006; Bretscher, 2009; Baccaglini-Frank & Mariotti, 2010; Hegedus & Moreno-Armella, 2010; Gómez-Chacón, 2012). From this point of view, it may be described that the students need to develop instrumented techniques and utilization schemes with the DGE, in order for it to become a personalized instrument where exploration for invariants can occur in, primarily, conjecturing activities (e.g. Baccaglini-Frank & Mariotti, 2010). What do such utilization schemes (and corresponding instrumented techniques) entail?

The technique of exploring figures for invariants by dragging presumes that the students are aware of the relationship between the elements of a figure which determine the outcome of a dragging action, corresponding to van Hiele levels 2–3 (vH lvls 2–3) (Hölzl et al., 1994).

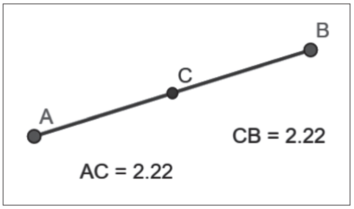

Figure 4. *Midpoint C is locked in the relationship AC = CB*

For example, understanding why the midpoint C in line segment AB will move when the free points A or B are dragged in figure 4, and why it is not possible to drag point C. This requires an awareness of the theoretical properties of figures which are mediated perceptually by the DGE (vH lvl 2). Therefore, the dragging technique/scheme to explore for invariants involves moving between the spatiographical and theoretical levels (Laborde, 2005b) which students need to coordinate. The spatiographical level refers to the perceptual appearance of a figure, while the theoretical level refers to the theoretical properties of a figure. Arzarello et al. (2002) describe how a DGE can potentially link the spatiographical level to the theoretical level in ascending and descending processes (vH lvl 1–2). Ascending happens when the students shift from the perceptual level to the theoretical level, while descending happens when the students shift from theory to perception. For example, if a student has made a conjecture about the theoretical property AC = CB in the construction in figure 4, she might validate it by a dragging test. The theoretical conjecture is confirmed perceptually in a descending process. On the other hand, if a student is unaware of the theoretical properties, the activity of dragging may prompt a shift towards awareness of the theoretical properties, since the properties are mediated perceptually by the DGE in the form of invariants (the property AC = CB remains). The perceptual output may result in theoretical awareness in an ascending process. Arzarello and colleagues (2002) found that students exploring geometrical figures by dragging in DGEs shift back and forth between empirical and deductive reasoning in ascending and descending processes. Many studies have found that DGEs can support students in connecting empirical and theoretical mathematics (e.g. Lachmy & Koichu, 2014, Baccaglini-Frank & Mariotti, 2010; Mariotti, 2006; Guven et al., 2010; Hadas et al., 2000; Jones, 2000; Laborde, 2005b). Even though DGEs can present opportunities for students working on the proving process (Laborde 2000; Olivero, 2002; de Villiers 2004; Sinclair & Robutti 2013), particularly in the production of conjectures, DGEs can also present challenges because of the strong link to the spatiographical level (Sinclair & Robutti, 2013). From the perspective of the Theory of semiotic mediation, the students' personal meanings underlying the initial situated signs stemming from the DGE activity do not necessarily relate to the theoretical aspects of the DGE constructions. However, the teacher can mediate the evolution of mathematical meanings and mathematical signs, which is needed if the students are to notice theoretical relationships in the DGE activity (vH lvl 2–3).

Additionally, comprehension of direct and indirect invariants is required. The students need to understand the difference between invariants caused by the construction and invariants caused by the rules of Euclidean geometry in order to investigate a construction to make conjectures (see example in next section) (vH lvl 3). Furthermore, exploration of invariants requires capacity regarding certain dragging techniques/schemes. Research on ways of dragging in DGEs has resulted in a classification of several dragging modalities, which can broadly be divided into two categories (Hölzl, 2001; Leung, 2015): (1) Dragging for searching/discovering, containing dragging modalities where the student drags in order to explore the figure for new properties. For example: *wandering dragging* – dragging randomly to try to discover regularities or interesting configurations; *guided dragging* – dragging basic points to make a particular shape; *maintaining dragging* – realizing an interesting configuration and trying to keep the specific property invariant while dragging (noticing a soft invariant); and (2) Dragging for testing, encompassing the dragging modalities in which the students drag to test an expected reaction from the construction. For example, *the dragging test* – dragging objects in order to see if the construction maintains desired properties, i.e. if it is robust; *the soft dragging test* – testing a conjecture about a soft invariant (Arzarello et al., 2002; Baccaglini-Frank & Mariotti, 2010). Similarly, measuring modalities for searching and testing have been classified into two broad categories: measuring for discovery – wandering measuring, guided measuring, perceptual measuring; and measuring for testing – validation measuring, proof measuring (Olivero & Robutti, 2007). Students' development of instrumented techniques and utilization schemes for dragging and measuring to explore, develop and test conjectures is a prerequisite for working on tasks that can support students' progression of RC. In addition, the development of schemes and techniques for utilizing tracing and the feedback function of DGEs can be valuable in order to work on tasks that may mobilize students' RC, which will be explained in the next section.

Task design

The literature review revealed several types of task design in DGEs. Some studies report on task design principles or models for task design (e.g. Lin et al., 2012; Komatsu & Jones, 2018; Fahlgren & Brunström, 2014; Olsson, 2017), while some have developed models to assess task quality (e.g. Trocki, 2014; Trocki & Hollebrands, 2018) in relation to DGEs in general, and with focus on reasoning and proof (Baccaglini-Frank et al., 2013, 2017, 2018; Leung, 2011; Sinclair, 2003). Models for task design will not be introduced in this article, but using the perspective of RC, five types of task design were identified as having the potential of mobilizing students' development of different characteristics of RC, thereby potentially increasing students' *degree of coverage* of RC.

Construction tasks. (1) The students can be supported in creating and justifying mathematical claims in general by offering tasks similar to what Mariotti (2012) coined "construction tasks", which require the students to construct robust figures with specified invariants using limited construction commands. Such a task could involve constructing a robust square using only construction commands such as points, line segments, lines, perpendicular lines (some might prefer not to allow this command), circles, and intersection points (see figure 5). The students have to describe the procedure and explain why the figure remains invariant, which is to create and justify a mathematical claim in terms of the RC. Dragging is an instrument to confirm the validity of the construction. This type of

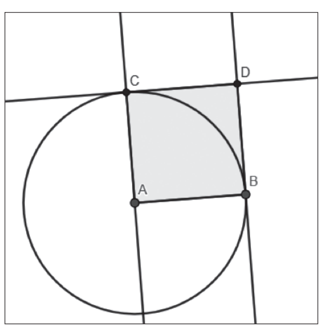

Figure 5. *Constructing a robust square using specified construction commands*

task can help students to develop an awareness of direct invariants, and is therefore a valuable prelude to working on conjecturing open tasks, which is described next.

Direct indirect invariants. (2) If a student has developed an awareness of the constructed direct invariants, then the discovery of indirect invariants through dragging can lead to conditional "if-then" conjectures, with the direct invariants being the premise for the indirect invariants (Baccaglini-Frank & Mariotti, 2010, Lachmy & Koichu, 2014). For example, a conjecturing open task for students could be to construct ΔABC, the midpoints of two of the sides, and to draw a line segment connecting these midpoints (figure 6).

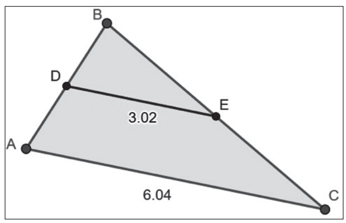

Figure 6. *Direct and indirect invariants in a robust construction*

In exploring the construction by dragging free points, students can discover the invariant parallelism of DE and AC, which was not a property of the initial construction. The potential of measuring can also be used to discover the invariant relationship of length DE being half of length AC. These conjectures can, with guidance from the teacher (discussed later), lead to a proof process of the midpoint theorem. In this case, dragging and measuring are tools to investigate the theoretical properties of the figure.

Maintaining dragging. (3) The usage of robust constructions has been prevalent in DGE teaching (Ruthven et al., 2005), but in a slightly different type of task, the students can be prompted to make a soft construction and to try to discover the conditions for which some property is maintained, using the maintaining dragging modality (Baccaglini-Frank & Mariotti, 2010). For example, a simple task for students could be to construct line segments AB and BC and look for the positions of B which satisfy AB = BC, using trace activated on point B. By interpreting the trace path shown in figure 7, the students can discover and perhaps conjecture that points which are equidistant from two given points would all lie on the perpendicular bisector of the line segment joining the two points. In this case, the potential of tracing is utilized to unveil an underlying invariant.

Pseudo-objects. (4) By offering tasks instigating students to construct non-constructible pseudo-objects (Baccaglini-Frank et al., 2013, 2017

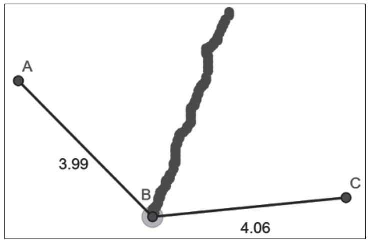

Figure 7. *Maintaining the property AB=BC in a soft construction with trace activated*

2018), abilities in proving by means of contradiction can evolve. A pseudo-object contains contradictory properties with regard to Euclidean theory, and it is therefore not possible to construct in a DGE. As students attempt to construct the pseudo-object, (e.g. figure 8 or 9) the feedback affordance provided by DGEs can assist the student in realizing the impossibility of the construction. Designing such tasks involves identifying proto-pseudo objects, which are objects that have the potential of becoming pseudo-objects for the students, for example a triangle in which two angle bisectors are perpendicular (Baccaglini-Frank et al., 2013, 2017, 2018).

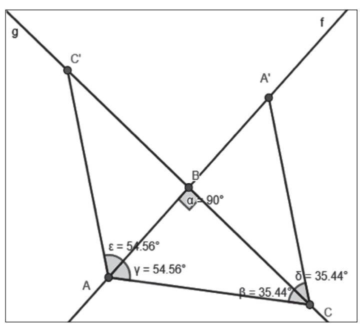

Figure 8. *Starting from two perpendicular lines g and f (the angle bisectors) and reflecting AC in them*

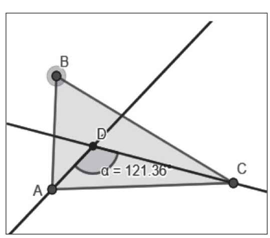

Figure 9. *Investigating the pseudo-object by constructing the triangle first and then dragging and measuring*

Hidden conditions. (5) Another aspect of RC is to understand the role of counterexamples. To this end, the potential of dragging can be utilized by designing tasks, which prompts students to discover counterexamples to conjectures as they manipulate constructions. The conjectures may be given in the task description or discovered by the students themselves. Komatsu and Jones (2018) suggest that such heuristic refutation tasks could include ambiguous diagrams with "hidden conditions", exemplified in figure 10 with the accompanying task: "there are four points A, B, C, and D on circle O. Draw lines AC and BD, and let point P be the intersection point of the lines. What relationship holds between ΔPAB and ΔPDC? Write your conjecture. (2) Prove your conjecture." (Komatsu & Jones, 2018, p. 9). The students might argue that $\angle$ BPA = $\angle$ DPC (vertical angles are equal) and that $\angle$ ABP = $\angle$ PCD (inscribed angle theorem), hence $\angle$ PAB ~ $\angle$ PDC.

But when the students are prompted to drag points A, B, C and D, they might discover local counterexamples to the conjecture, such as figure 11, and be motivated to revise their conjecture.

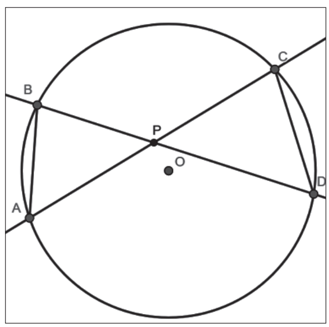

Figure 10. *Diagram with "hidden conditions," inviting insufficient conjectures*

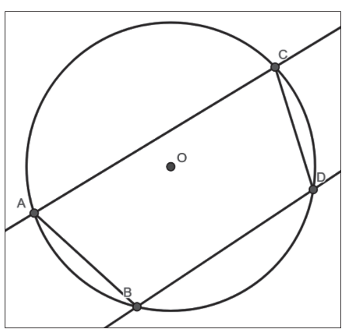

Figure 11. *Counter-example to the conjecture, AC parallel to BD*

4.6 Role of the teacher

The teacher plays an important role in helping students to transform personal meanings into mathematical meanings as described by Bartolini-Bussi and Mariotti (2008). To do so, the teacher should be conscious of the mathematical goal of the DGE class activity, and should use opportunities to emphasize mathematical meanings of the DGE-mediated signs. When teaching, discussing or giving feedback, the teacher should identify emerging signs, and try to inject mathematical meaning into these signs (Bartolini-Bussi & Mariotti, 2008). The teacher can guide the students to understand the mathematical implications of the feedback provided by the DGEs as well as the mathematical meaning of dragging, measuring, and tracing. As the students produce initial situated signs and develop personal meanings in relation to feedback, dragging, measuring, and tracing, the teacher can mediate the evolution of mathematical meanings and signs. RC includes the ability to present formal arguments and to develop arguments based on heuristics into formal proofs (Niss & Højgaard, 2011). In order to further progress the students' *degree of coverage*, the teacher needs to engage the students in proving their conjectures. This, however, does not happen automatically. Some studies (e.g. Marrades & Gutiérrez, 2000; Connor et al., 2007) even suggest that exploratory use of DGEs can inhibit the progression of students' deductive proving, since the students are empirically convinced that a fact is evident, and do not see the point of having to prove it (again). However, studies also show that DGE exploration does not have to jeopardize this progression (Lachmy & Koichu, 2014; Sinclair & Robutti, 2013). Researchers highlight the important role of the teacher in guiding and motivating the students towards justifying their conjectures and towards applying a theoretical approach (e.g. Mariotti, 2012; Arzarello et al., 2002). De Villiers (2007) argues against questioning the conviction that empirical methods give, or trying to convince students to undertake theoretical verification to further verify what they already find evident, but stresses that instead the teacher should motivate the students by asking "why" the fact is evident. Trocki (2014) argues that motivation and guidance for students to connect informal exploration and conjecturing to theoretical justification might also be incorporated into the DGE task itself, for instance by including questions that prompt students to justify their conjectures.

The possession and development of RC in relation to DGEs

Based on the review results presented in chapter four, we can analyze how the development of RC may be supported with DGEs using the terminology in the KOM framework concerning a person's attainment of a competency. As previously mentioned, the attainment is described in three dimensions: *degree of coverage*, *radius of action*, and *technical level*.

Degree of coverage

It is evident from the findings that the development of students' possession of various characteristics of the RC can be supported by utilizing DGE potentials (task types 1–5), thereby increasing the students' *degree of coverage* of RC, in particular the ability to create and present informal arguments, because the dragging arguments are empirically based on invariant relationships discovered in many examples. Hence, focus is primarily on the exploration and conjecturing phase of the proof process. For example, the direct/indirect invariants task (task type 2) illustrates how dragging and measuring might be utilized in the conjecturing phase of conditional statements. Similarly, the soft invariant task illustrates how dragging, measuring, and tracing can be utilized in exploring and conjecturing about equidistant points and the perpendicular bisector of the line segment joining them. However, with guidance from the teacher, the students' abilities in relation to the second phase of the proof process – validating the conjecture – can also evolve, thereby expanding the students' *degree of coverage* of RC. While validating the conjecture, the students might primarily work outside the DGE, but they are likely to return to verify/refute their progress as found by Olivero and Robutti (2007). As mentioned, RC consists not only of being able to reason yourself, but also of being able to follow and evaluate reasoning put forward by others (Niss & Højgaard, 2011). This may be incorporated in the teaching design, in order to support the students' development of their *degree of coverage* of RC, by organizing the teaching environment in a way that encourages the students to collaborate (e.g. to work in pairs). It can also be explicitly addressed in the tasks themselves, e.g. by asking the students to explain their reasoning to each other, in pairs and in class discussions. Additionally, they can be required to evaluate the reasoning put forward by others. When the students manage to validate their conjectures deductively, the teacher (or perhaps prompts in the task) should highlight the difference between the theoretical proof and the conjecture to foster an understanding of the difference between the two – which is a characteristic of the RC.

Radius of action

As the students work on the aforementioned types of tasks (1–5), their *radius of action* regarding RC may expand, because they can progressively activate their RC in an increasing amount of subject matter. Covering a variety of areas in geometry in the task design will further this agenda. Furthermore, the students' radius of action may be promoted, since they develop the ability to activate their RC in the context of using a DGE for this purpose.

Technical level

The students' progression within the *technical level* of RC can also be supported, particularly if consecutive tasks demand higher levels of reasoning from the students, for example by successively increasing the number of steps needed in the chain of reasoning to solve the task or the number of presuppositions needed to prove a conjecture. As previously mentioned, the degree of difficulty regarding the technical level of RC involved in solving a certain task has a subjective character, as it depends on the educational level of the students trying to solve the task. The guidelines do not address a particular educational level; however, if we take Danish lower secondary school as an example and look at the task example used to characterize task type 3, then we can imagine that the task is not too demanding with regard to students' *technical level* of RC, while the example in task 2 is more complicated, even though the conjecturing phase of the example in task 2 may be relatively simple, proving the conjecture involves a few steps and demands a higher degree of *technical level* of RC.

Formulating guidelines for teaching with DGEs to support RC

Based on the findings from the review on DGE literature and the subsequent analysis (auxiliary questions (i) and (ii)), guidelines are suggested in table 1 in appendix A. In essence, the guidelines are an analysis of the semiotic potential of DGEs when the educational aim is to support students' development of RC, with the analysis building on previous research in the field.

There are six columns in the table; the first column holds steps of progression 0–4, in which steps 2–3 have a subset of steps. The van Hiele levels of mathematical thinking are also indicated in the first column. Although the guidelines are presented in steps of a hierarchical nature, and van Hiele's levels are used, they are not considered discrete or clearly continuous, which can be seen with some overlapping descriptions in steps 1 and 2 (a, b, c, and d). In addition, it may well be that the development of several steps can occur at the same time, for example developing an understanding of free and locked objects (step 1) at the same time as developing basic DGE proficiency (step 0). Columns two and three address the dimension of students' cognition using the notions of instrumented techniques and utilization schemes[6]. Column four indicates what kind of tasks might mobilize the desired techniques and schemes, while column five describes the role of the teacher in facilitating the process. Column six describes which characteristics of the RC the DGE-mediated activity is expected to mobilize.

Below, some comments are added to each step of progression.

0 Basic DGE proficiency regarding commands of construction and measuring is needed to work on tasks which can support the development of RC.

1 Being able to discern between free and locked objects requires that the students are aware of the theoretical properties of figures (vH lvl 2). Additionally, an awareness of theoretical relationships between the elements of a figure or between figures is required (vH lvl 3).

2 (a, b, c and d) Awareness of the hierarchy of dependencies which determine the dragging outcome is necessary. This covers an understanding of free points, direct invariants, and robust and soft constructions (vH lvl 2–3). Furthermore, the ability to discern between direct and indirect invariants is required in, for example, conditional "if-then" conjecturing (vH lvl 3–4). Comprehension of certain dragging modalities and measuring modalities can support the exploration for conjectures. Tasks which support this cognitive development and the *degree of coverage* of RC include: construction tasks that encourage creation and assessment of mathematical claims, as well as understanding of direct invariants (vH lvl 2–3); conjecture open tasks which encourage construction (robust and soft) of direct invariants that bring on indirect invariants and allow for exploratory work in order to support the development of the first phase of the proof process (vH lvl 3–4).

3 (a and b) Understanding and being able to exploit the feedback function inherit in the DGE to investigate the construction of non-constructible pseudo-objects in order to foster abilities in proving by means of contradiction (vH lvl 3–4); understanding and being able to exploit the feedback function inherit in the DGE to find counterexamples to conjectures about diagrams with hidden conditions (vH lvl 3–4).

4 Being able to prove the conjectures. The role of the teacher is important in motivating the students towards theoretical validation of their conjectures to develop the second phase of the proof process (vH lvl 4). The teacher can encourage the students to return to the DGE in order to verify/refute their progress as they are proving their conjectures.

Concluding remarks

The process of answering the research questions comprised of searching the literature for DGE affordances that are considered potentials in relation to supporting students' development of RC, and of identifying which dimensions the guidelines should entail. Four DGE potentials were identified: feedback, dragging, measuring, and tracing. The utilization of these was described in three dimensions of the guidelines: students' cognition, task design, and role of the teacher. The guidelines in Appendix A contain five steps of progression, in which the dimensions are addressed and the expected mobilization of RC is described.

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Notes

1 The English translation of the acronym is *Competencies and mathematical learning*. The KOM framework was published in Danish in 2002 and in English in 2011. In 2019, an updated version of the framework was proposed by the authors (Niss & Højgaard, 2019).

2 https://www.zentralblatt-math.org/matheduc/

3 https://eric.ed.gov/

4 Congress of the European Society for Research in Mathematics Education.

5 To clarify, the notion of ”potential” is used here regarding affordances of DGEs which are not easily available in other common mathematics education mediums, in particular the pencil and paper environment. It could be argued that the possibility of constructing, for example, a circle or a regular polygon is an affordance of DGEs, but it is not a major addition compared to the pencil and paper environment, and therefore not considered a potential in this review.

6 The table provides a broad description. A fine-grained analysis is needed at the level of schemes, e.g. rules of action, operational invariants etc. (Vergnaud, 2009).

Appendix A

Table 1. *Research-based guidelines for mathematics teaching with dynamic geometry environments to support students' development of reasoning competency.*

Steps of progression	Guidelines				Aim
	Students' cognition		Tasks	Teacher's role	Reasoning competency
	Instrumented techniques	Utilization schemes		(during instruction, discussions and feedback) *Generally*: Identifying emerging signs and progressing towards mathematical meanings.	*Degree of coverage*: which characteristic is activated.
0. Basic DGE proficiency, van Hiele levels 1–2.	Students need to develop technical proficiency with the DGE. They need to be able to use the different commands for constructing, measuring, and dragging.	Technical and mathematical knowledge about the commands of constructing, measuring, and dragging.	Tasks which require the students to use the commands of the DGE. E.g. constructing points, lines, line segments, parallel lines and circles, finding intersection points, measuring angles, lengths, areas etc. and dragging these objects.	Highlighting the mathematical meaning in DGE-mediated activities, e.g. constructing a circle with two mouse clicks (definition of a circle), or the mathematical meaning of dragging as a way of varying the existing construction, such as varying coordinates of a point.	Prerequisites for working on conjecturing tasks and for justifying mathematical claims.
1. Free and locked objects, van Hiele levels 2–3.	Making constructions with free and locked objects. Dragging/measuring to investigate the construction.	Understanding the difference between free and locked objects. This requires attention to and comprehension of the theoretical properties of constructions.	Construction tasks which highlight the difference between free and locked objects. E.g. constructing two points and a midpoint between them. Describe why you can/cannot drag some points.	Focus on the mathematical relationship between elements of the construction which determine whether the objects are free or locked, which means focus on the theoretical aspects of figures, e.g. the mathematical meaning of a midpoint.	Justify mathematical claims. Prerequisite for working on conjecturing tasks.
2a. Robust constructions, van Hiele levels 2–3.	Making robust constructions with dependencies between elements in the construction, so that some desired properties remain invariant when free objects are dragged. Dragging/measuring to investigate the construction.	Understanding that direct invariants occur because of the theoretical properties induced in the construction.	Tasks that require the student to construct figures in which some properties remain invariant during dragging. ”Construction tasks” (Mariotti, 2012). E.g. constructing a quadrilateral with one right angle.	Highlighting and encouraging student focus on the theoretical properties of the figure. E.g. right angles require perpendicular line segments. Invariants occur because of the mathematcal relationship between elements of the figure.	Justify mathematical claims. Prerequisite for working on conjecturing tasks. Exploring and conjecturing.
2b. Soft constructions, van Hiele levels 2–3.	Making soft constructions with non-dependencies between some elements in the construction, so that some desired properties remain invariant only when certain conditions are satisfied. Dragging/measuring and tracing to investigate the construction.	Understanding that direct non-invariants occur because of the lack of dependencies between theoretical properties induced in the construction.	Tasks which require the student to construct figures with soft invariants, in which some properties can be maintained only under certain conditions. E.g. finding the positions of point B for AB = BC.	Highlighting and encouraging student focus on the theoretical properties of the figure. E.g. meaning of perpendicular bisector. Non-invariants occur because of the mathematical relationship between elements of the figure.	Justify mathematical claims. Exploring and conjecturing.

2c. Dragging/ measuring modalities for exploration, van Hiele levels 2–3.	Being proficient in different dragging/measuring modalities: Dragging for searching/dragging for testing; measuring for searching/ measuring for testing.	Understanding fruitful dragging and measuring modalities in order to explore constructions to unveil indirect invariants and make conjectures. Including the "maintaining dragging" modality for soft invariants.	Tasks requiring the students to make conjectures by dragging and measuring in certain ways in order to notice indirect invariants. This includes tasks with soft invariants.	Explaining and illustrating how to drag free points with different aims: Randomly, looking for invariance of properties and measures, maintaining a property (maintaining dragging).	Justify mathematical claims. Prerequisite for working on conjecturing tasks. Exploring and conjecturing.
2d. Direct and indirect invariants, van Hiele level 3.	Constructing direct invariants which induce indirect invariants because of Euclidean theory. Dragging/measuring to investigate the construction.	Understanding the difference between direct and indirect invariants, and the connection between them.	Tasks requiring the students to make constructions with direct invariants, where dragging free points in the construction also unveils (surprising) indirect invariants.	Explaining and highlighting that direct invariants can induce indirect invariants because of the "rules of Euclidian geometry". E.g. lines perpendicular to parallel lines are parallel. Introducing the "if-then" relationship between direct and indirect invariants. Stressing the empirical nature of the conjecture.	Justify mathematical claims. Exploring and conjecturing.
3a. Feedback: non-constructible pseudo objects, van Hiele levels 3–4.	Constructing, measuring and dragging to test the possibility of an object.	Understanding the feedback function inherent in the DGE, and thereby the possibility of exploring whether objects can be constructed.	Tasks instigating the students to construct non-constructible pseudo-objects.	Highlighting and encouraging student focus on the theoretical properties of the non- constructible figure. Injecting mathematical meaning into the students' evolving signs of conflictions regarding the object.	Exploring and conjecturing. Abilities in proving by means of contradiction can be developed.
3b. Feedback: Counterexamples to conjectures, van Hiele levels 3–4.	Constructing figures and exploring (dragging, measuring) in order to find counterexamples to conjectures.	Understanding the feedback function inherent in the DGE. Understanding how a counterexample forfeits the conjecture.	Tasks which prompt students to discover counterexamples to conjectures as they manipulate constructions. Such tasks could include diagrams with hidden conditions, and the tasks should explicitly prompt the students to find counterexamples.	Highlighting and encouraging student focus on the theoretical properties of the figure which underlie the conjecture to which counterexamples are to be found. Injecting mathematical meaning into the students' evolving signs regarding the object.	Exploring and conjecturing. Understanding the meaning and role of counterexamples.
4. Proving the conjectures from steps 2b, 2c, 2d, 3 and 4, van Hiele level 4.	Using DGEs to verify/refute progress on proving the conjectures.	Understanding the feedback function inherent in the DGE, and thereby the possibility of verifying/ refuting conjectures.	Follow-up tasks requiring the students to prove their conjectures. Students can work outside the DGE, but return to verify/refute their progress.	Motivate the students to undertake theoretical verification by asking "why" their conjecture is true instead of dismissing the empirical evidence provided by the DGE (De Villiers, 2007).	Abilities in proving; developing an argument based on heuristics into formal proof. The difference between a proof and other forms of mathematical reasoning such as explanations based on examples.

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Paper II

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Teachers Reporting on Dynamic Geometry Utilization Related to Reasoning Competency in Danish Lower Secondary School

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Abstract
The article describes the development and analysis of a quantitative study investigating to what extent the potential of dynamic geometry environments (DGEs), in relation to mathematical reasoning competency, are utilized in lower secondary schools in Denmark (grades 7–9). The study entails a questionnaire, which was developed on the basis of an extensive review that uncovered four potentials of DGEs in relation to reasoning competency: feedback; dragging; measuring; tracing. 220 Danish lower-secondary mathematics teachers completed the questionnaire. Analysis indicates that the potentials of measuring and dragging are utilized to some degree, feedback to a lesser degree, while tracing is almost non-existent. Furthermore, there are signs that DGEs are used as a substitute for the paper-and-pencil environment to solve tasks that were originally designed for paper and pencil. Possible improvements of praxis are discussed and the integration of the results into praxis in further research is elaborated upon.

Keywords Dynamic geometry environments · GeoGebra · Reasoning competency · Survey

Digital technologies are widely implemented in Danish primary and lower secondary schools, in part due to heavy investment from the Ministry of Education over the last couple of decades (e.g. Undervisningsministeriet 2015). Consequently, the availability and usage of digital technologies has become commonplace in mathematics education at all levels in Denmark. In primary and lower secondary school, the dynamic geometry software GeoGebra is particularly popular. This can be considered a positive outcome, since many studies highlight the affordances of dynamic geometry environments (DGEs hereafter), as potentials in supporting students' development of mathematical

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reasoning, conjecturing and in proving (e.g. Jones 2000; Leung 2015; Edwards et al. 2014).

In the optic of the Danish competency framework (KOM)[1] (Niss and Jensen 2002; Niss and Højgaard 2011, 2019), these potentials are related to reasoning competency. However, it seems that the manner in which the affordances are utilized (if they are) is essential (Jones 2005). The motivation behind this study stems partly from an underlying assumption that the potentials of DGE, such as GeoGebra, are not made use of in Danish lower secondary school, even if the software is indeed popular. The assumption is that DGEs are predominantly used as a substitution for the paper-and-pencil environment.

No previous quantitative research exists regarding how teachers actually incorporate DGEs in Danish lower secondary schools. I could not find quantitative results on the matter internationally either. However, qualitative studies on the use of technology in mathematics education have shown that teachers refer to the added pace and productivity provided by technology (Ruthven et al. 2004.) In Bozkurt and Ruthven's (2017) case study of an expert secondary school teacher, one who was "recognised as having successfully developed and integrated use of GeoGebra in his classroom practice" (p. 312), the teacher referred to the crucial affordances of GeoGebra as being accuracy and speed, as well as dragging and the feedback of the environment allowing for discovery-oriented tasks.

In order to gain insight into this area, the study described in this article aims to investigate how DGEs are actually used in lower secondary school in Denmark, with particular interest in their potential in relation to *reasoning competency*, by posing the following research question:

> *To what extent do Danish lower secondary school (grade 7–9) mathematics teachers utilize the potentials of DGEs, in relation to mathematical reasoning competency, in their classroom?*

The problem is investigated by analysing the results from a questionnaire, which was developed for Danish lower secondary school teachers. In this article, the development, results and analysis of the questionnaire is presented. Firstly, the research question calls upon an elaboration of what mathematical reasoning competency constitutes, as well as what is meant by potentials of DGEs. This is on the agenda in the following section.

Theoretical Constructs for Developing the Questionnaire and Analysing the Results

Reasoning competency (RC hereafter) is one of eight mathematical competencies in the KOM framework (Niss and Jensen 2002; Niss and Højgaard 2011, 2019), which introduces concepts to describe what mathematical mastery entails across mathematical

[1] KOM is an acronym for "Kompetencer og matematiklæring", which translated to English means "Competencies and mathematics learning". It was first published in Danish (Niss and Jensen 2002) and subsequently translated into English (Niss and Højgaard 2002/2011). In 2019, *Educational Studies in Mathematics* published an article from the authors with minor revisions to the framework. Note that 'Jensen' and 'Højgaard' is the same author whose surname has changed between publications.

subject matter and educational level in a competency-based approach. The framework has influenced the curriculum at all levels in mathematics education in Denmark. In the lower secondary school curriculum, *mathematics common goals*, learning goals are described both in terms of the expected subject matter knowledge and student possession of the mathematical competencies (Børne- og Undervisningsministeriet 2019). Rasmussen et al. (2017) found that lower secondary school mathematics teachers do use and adhere to the common goals.

The KOM framework has also influenced mathematics education in other parts of the world (Niss et al. 2016), for instance, by shaping the foundation of the PISA assessment and analytical framework for mathematics (OECD 2017). In what follows, the main constituents of RC are elaborated upon: for a full account, the interested reader is directed to Niss and Højgaard (Niss and Højgaard 2002/2011).

The core of RC constitutes the ability to create and carry out both informal and formal arguments in order to justify mathematical claims, as well as to be able to follow and evaluate such argumentation put forward by others. Additionally, it entails understanding what a mathematical proof is, including the role of counter-examples. It also comprises the ability to discern proof from other forms of mathematical reasoning, such as example-based explanations and being able to develop such arguments into formal proof. Furthermore, it is about creating and justifying mathematical claims in general, such as answers to questions and problems (Niss and Højgaard 2002/2011).

As argued in Højsted (2019a), the notion of proof, in the context of school mathematics, may be understood as a process which involves exploring, conjecturing and justifying, and not merely theoretical validation of already-stated theorems (Arzarello et al. 2002, 2007; Mariotti 2012; NCTM 2008). In part, this may be due to the facilitation to experiment and investigate provided by digital technologies such as DGEs (Sinclair and Robutti 2013). The understanding of proof as a process resonates with the abilities to investigate and do mathematics described in the KOM framework. Therefore, it is argued, that RC also entails exploration and conjecturing (Højsted 2019a).

Potentials of DGEs in Relation to RC

In order to formulate questions that investigate to what extent the potentials of DGEs in relation to RC are being utilized in lower secondary school, it is first necessary to describe what these potentials are. To this end, the results from an extensive review into the potentials of DGEs in relation to supporting students' development of RC were used (Højsted 2019a, 2019b). The meaning of 'potential' in this context is affordances of DGEs, which are not available in other typical mathematics education tools, such as paper and pencil.

Four RC potentials were extrapolated in the synthesis of the review: feedback; dragging; measuring; tracing (Højsted 2019a). These are not separate domains; dragging especially intervenes in all three other potentials. It is particularly in relation to conjecturing that they were deemed to be potentials. A short elaboration of the potentials is presented here.

Since DGEs mimic theoretical systems, typically Euclidean Geometry, they offer an environment in which only constructions that abide by the rules of the theoretical system can be constructed (Balacheff and Kaput 1997). Therefore, the DGE inherently provides *feedback* to the user, e.g. by not allowing for imprecise measurements or constructions that contradict Euclidian theory. Figures constructed in the environment can also be manipulated dynamically by *dragging*. The elements of a dynamic figure are locked in a hierarchy of dependencies, which decide the outcome of a dragging action (Hölzl et al. 1994). The dependencies are in fact the theoretical properties of the figure, which are decided by the construction method and by the theory of Euclidean Geometry governing the system. These properties remain invariant during dragging, which allows for discovery of the theoretical properties of constructions in Euclidian Geometry. Therefore, dragging in DGE can link the perceptually mediated appearance of a figure to the theoretical properties of the figure, which Laborde (2005b) refers to as moves from the spatio-graphical level to the theoretical one.

Arzarello et al. (2002) classified different ways of dragging in DGEs, which can broadly be divided into two categories: dragging to search/discover and dragging to test (Leung 2015). 'Robust' constructions are those that resist to the so-called 'dragging test' (Arzarello et al. 2002), which means that the properties of the figure are conserved during dragging. On the contrary, in 'soft' constructions not all properties are conserved (Healy 2000; Laborde 2005a). Dragging may be performed on a soft construction in order to discover the conditions for which some property is maintained – using the maintaining dragging modality (Baccaglini-Frank and Mariotti 2010).

Beginning with *The Geometer's Sketchpad*, most DGEs, including GeoGebra, contain *measuring* tools, which can be used to find the measures of various objects in a construction. When figures are manipulated dynamically, the measurements are updated instantly. Students may also use the maintaining dragging modality (Baccaglini-Frank and Mariotti 2010) to fix the value of a measurement and thus determine whether a property is invariant. Additionally, the possibility of *tracing* an object during dragging offers the possibility of visualizing underlying invariant relationships of the construction, for instance in soft constructions in which a property is maintained when dragging is performed in a particular manner (Baccaglini-Frank and Mariotti 2010).

In the analysis of results, utility is found in Artigue's (2002) distinction between *pragmatic* and *epistemic* values of techniques. In this optic, a technique is a way of solving a task, and the pragmatic value of a technique refers to the productive potential of said technique, e.g. how efficient it is or to how many situations to which it may be applied. The epistemic value concerns the manner in which the technique promotes understanding of the mathematical object at hand.

Method

Developing the questionnaire consisted of a careful back-and-forth process of formulating questions that would investigate whether the DGEs' potentials related to RC were being utilized, while at the same time being formulated concisely and clearly to lower secondary school teachers. A web-based questionnaire was developed, which consisted of nine multiple-choice questions, in which a five-point Likert scale was used, as well as one open-ended question and some background information questions.

The questionnaire was distributed through two platforms; (1) a link for self-enrolment to the questionnaire was sent through the email list of the Danish national maths counsellor network with participation from 97 of the 98 municipalities of Denmark; (2) a link for self-enrolment was posted on two popular Facebook groups for Danish lower secondary school teachers, "GeoGebra Hangouts" and "We who teach in lower secondary school". In order to combat the usual problem of low participation response and completion in web-based questionnaires, steps were taken with regard to design and language (Fan and Yan 2010), and a monetary incentive was included (Göritz 2010) in the shape of a lottery with a prize of DKK 4000,- to a single winner. Respondents were required to add their names and email in order to take part in the lottery. This also gave the opportunity to check for double entries.

GeoGebra is by far the most popular DGE in lower secondary school in Denmark. Consequently, it was decided, for the sake of clarity, to formulate the questionnaire directly towards GeoGebra usage instead of more general 'DGE usage'. However, GeoGebra is a system that includes not only the geometry environment, but also CAS, a spreadsheet, etc. Therefore, in every question it was explicitly specified that the question was related to the 'geometry part' of GeoGebra. In the analysis of the data, results are presented from multiple-choice questions in the form of frequency tables and then analysed. The method used to analyse the open-ended question is presented in the open-ended question section.

Results and Ensuing Analysis

220 lower secondary school teachers completed the questionnaire.[2] Results and analysis from one background question (Q0), nine multiple-choice questions (Q1–9) and the open-ended question (Q10) are presented here. This section is divided into four sub-sections, the first three corresponding to the three types of questions mentioned.

Background Question Q0: The Extreme Population of the Study

Albeit not directly related to the research question, it is relevant to mention the results from question Q0, which was a background question, in order to shed some light on the population of the study. Since the teachers participating in the questionnaire are not a representative sample of the entire population, it is pertinent to consider what kind of data sample we have at hand. Even though we cannot conclude with certainty, certain arguments about the population may be put forward.

Results from Q0 (Table 1) show that the respondents frequently include GeoGebra in their mathematics classroom practice.

Based on the results from Q0 (in combination with the fact that the questionnaire was among other means distributed on the Facebook group "GeoGebra Hangouts", which essentially is a group for people interested in GeoGebra), and that participation in the survey was through self-enrolment, it is reasonable to infer that the majority of respondents in the survey are teachers who actually use GeoGebra regularly in their

[2] There are approximately 4000 mathematics teachers in grades 7–9 in Denmark.

Table 1 A background question on the rate of GeoGebra usage

$N=220$	Q0. How often do you and your students use the geometry part of GeoGebra in the mathematics class?
Every week	35.9%
Every other week	39.5%
Once a month	17.3%
Every other month	4.1%
Less	3.2%

mathematics classes. Perhaps it is even reasonable to assume that many of the respondents are teachers who are somewhat enthusiastic about GeoGebra.

Hence, it is important to consider that the respondents are not representative of the average Danish lower secondary school mathematics teacher population, but instead may be considered as a case of an extreme population (in regard to DGE usage). The results could therefore be interpreted in that light. For example, if it is found that the respondents in this population do not utilize the potentials of a DGE in relation to RC, then we may surmise that the average teacher does not either.

Multiple-Choice Questions Q1–9

Some questions were posed to find out which types of tasks the teachers give to their students: this was partly done in order to investigate the hypotheses that GeoGebra is merely used as a substitute for the paper-and-pencil environment. Partly, the results from these questions would also indirectly indicate whether the potentials are being utilized or not. For instance, if the students primarily work on tasks in GeoGebra, which were made for paper-and-pencil geometry, it is a strong indication that potentials such as dragging are not involved in the tasks, since dragging is not possible in paper-and-pencil geometry.

As can be seen from the results to Q1 (Table 2), a large proportion of the students seem to be working on such paper-and-pencil tasks, which indicates that the hypothesis holds some truth. However, the results from Q2 show that many teachers adapt paper-and-pencil tasks. Therefore, it cannot be ruled out that some teachers adapt the paper and pencil tasks in such a way that some of the four potentials, which are linked to the RC, are utilized.

Table 2 Q1 and Q2 regarding types of task

$N=220$	Always	Frequently	Occasionally	Rarely	Never	Don't know
Q1. Do students work on tasks in the geometry part of GeoGebra, which were originally made for paper-and-pencil geometry?	14.5%	38.6%	35.5%	9.1%	2.3%	0.0%
Q2. Do students work on tasks that were originally made for paper-and-pencil geometry, which you have adapted to be used in the geometry part of GeoGebra?	6.8%	34.1%	40.5%	11.4%	5.5%	1.8%

A premise for investigating theoretical properties of figures by dragging is that the students hold some understanding of the difference between free and non-free objects. Question Q3 (Table 3) aimed at finding out if students work on understanding this difference. It was expected that some teachers might not know what free and non-free objects were. Therefore, a short video was shown prior to Q3, along with an explanatory text, in which it was illustrated how two free points could be dragged, while their constructed mid-point could not. The predominant answers were *occasionally* (35.9%) and *rarely* (30.5%), which indicates that it is not something on which a lot of time is spent. Moves from the spatio-graphical level to the theoretical one (Laborde 2005b) are unlikely to occur if the students do not realise that the perceptually mediated appearance of the figure during dragging in DGEs is linked to the theoretical properties of the figure.

Results from Q4 show that more time is used on constructing robust figures. Considering that understanding the difference between free and non-free objects is necessary to construct them, this result is somewhat contradictory, but perhaps it is because the teachers discern between the implicit understanding needed to construct robust figures and time spent explicitly focusing on the difference between free and non-free objects, as two distinct activities. Q5 was aimed directly at finding out if *dragging* was being used as a means to investigate the theoretical properties of figures. An example was given of properties that can remain invariant under dragging (the medians of a triangle meeting at a point), because it was expected that some teachers might not understand the meaning of "investigating figures, to see which properties are maintained during dragging".

The teachers predominantly responded that their students *occasionally* (47.3%) or *frequently* (33.2%) engage in such activity, which are higher rates than in Q3 and Q4, and also a higher rate than expected beforehand. It is also somewhat surprising in light of the responses to Q2, since tasks which were originally made for a paper-and-pencil environment would not include prompts requiring dragging to investigate theoretical properties of figures. Nevertheless, the results do suggest that the potential of dragging is used regularly by students to investigate the properties of figures. It indicates that potentials of DGEs linked to the exploration and conjecturing part of the RC are utilized to some degree.

Table 3 Q3–Q5 concerning tasks focusing on theoretical properties and dragging

$N = 220$	Always	Frequently	Occasionally	Rarely	Never	Don't know
Q3. Do students work on understanding the difference between free objects and non-free objects in the geometry part of GeoGebra?	5.0%	13.2%	35.9%	30.5%	15.0%	0.5%
Q4. Do students work on constructing so-called 'robust' figures, i.e. figures that retain certain properties, when the free objects of the figure are dragged, in the geometry part of GeoGebra?	1.4%	24.1%	46.8%	17.7%	10.0%	0.0%
Q5. Do students work on investigating figures to see which properties are maintained during dragging in the geometry part of GeoGebra (e.g. that the medians of a triangle meet at a point)?	5.0%	33.2%	47.3%	10.9%	2.7%	0.9%

Table 4 Q6 and Q7 on measuring

$N = 220$	Always	Frequently	Occasionally	Rarely	Never	Don't know
Q6. Do students work on measuring figures in the geometry part of GeoGebra?	12.7%	66.8%	18.6%	0.9%	0.5%	0.5%
Q7. Do students work on measuring figures combined with dragging, in order to investigate how the measures change in the geometry part of GeoGebra (e.g. that the sum of angles in a triangle remains at 180°)?	5.9%	41.4%	43.6%	5.9%	3.2%	0.0%

Looking at results from Q6 and Q7 (Table 4), we can see that *measuring* takes place at a relatively high rate compared with dragging. In fact, it was also the highest rate compared with all other questions, also those that are not included in this article. In Q6, a full two-thirds of the teachers who responded reported that their students frequently work with measuring figures. Or course, Q6 does not reveal what sort of measuring is done. For instance, it could be the sort of measurement of figures which might as well be done with paper and pencil. However, Q7 gives more nuanced insight by asking more specifically about the measuring activity. The teachers report that their students frequently (41.4%) and occasionally (43.6%) work with measuring in combination with dragging in order to investigate invariant measurements.

One way of utilizing the *feedback* potential of DGEs is with tasks that instigate the students to construct non-constructible figs. Q8 (Table 5) reveals that 45.5% of teachers reported that their students occasionally work on such tasks, while 32.5% responded that their students rarely do so. Of course, there are also other ways of utilizing the feedback potential.

The results from Q9 indicate that possibility of *tracing* is unfamiliar to the teachers. The teachers mainly responded that their students rarely (34.1%) and never (45%) work with the trace command. It can be concluded that the potential of tracing to visualize underlying invariant relationships of constructions in conjecturing tasks that yield development of RC is yet to be utilized, for example in maintaining dragging tasks (Baccaglini-Frank and Mariotti 2010).

Table 5 Q8 and Q9 on non-constructible figures and tracing

$N = 220$	Always	Frequently	Occasionally	Rarely	Never	Don't know
Q8. Do the students work on tasks in the geometry part of GeoGebra where they are asked to try to construct figures which cannot be constructed?	0.9%	10.5%	45.5%	32.7%	10.5%	0.0%
Q9. Do the students work with the trace command in GeoGebra?	0.5%	1.4%	16.4%	34.1%	45.0%	2.7%

Open-Ended Question Q10

Since the goal of the questionnaire was to find out to what extent the four potentials were being used, the following open-ended question was put forward:

> *Q10: What do you think are the greatest affordances (if any) of the geometry part of GeoGebra*?

As mentioned, there were 220 lower secondary school teacher respondents in the survey; however, many of them provided several comments in their answers. These were separated into individual comments, giving a total of 357 comments to be analyzed. Hence, the individual response from many respondents included different comments that were assigned to different categories. In the examples of comments provided in this article, the full answers from the respondents are given, but the part of the answer that is in bold, is what caused it to be placed in the particular category, which is being exemplified. The other parts of the full answer may have been assigned to other categories.

One way of investigating the research question of the article was to count how often the four potentials were explicitly mentioned in the respondents' comments: no explicit mention of dragging, tracing or feedback was found in the comments. Seven comments explicitly included measuring as an affordance, but none of them was in relation to the dynamicity of measures, which allows for exploration of invariant measures. An example of a measuring comment is:

(i) *It* [GeoGebra] *is precise.* **It is able to calculate/measure (e.g. area/perimeter/angles etc.)**.

Even though the affordances are not mentioned explicitly, additional analysis and categorization of the comments can unveil some clues on whether or not the potentials are indeed utilized. Therefore, categories were developed by reading through the comments and developing categories to fit the characteristics of the respondents' answers. Each successive comment was classified either into previously developed categories or into a new category, which was developed ad hoc. Finally, some categories were merged, resulting in six categories, as shown in Fig. 1.

The most common type of comment belonged to the category 'Efficiency and precision' (27%). The comments in this category were characterized by their reference to the possibility of making constructions faster or more precisely in GeoGebra, presumably compared with paper and pencil. The measuring comment in example (i) was attributed to this category because of the mentioning of precision. Other examples are:

(ii) **Fast and precise drawings** ['drawings, a translation of *tegninger*, is often used synonymously with 'constructions' by lower secondary school teachers in Denmark].
(iii) **That you can quickly construct and measure figures**.

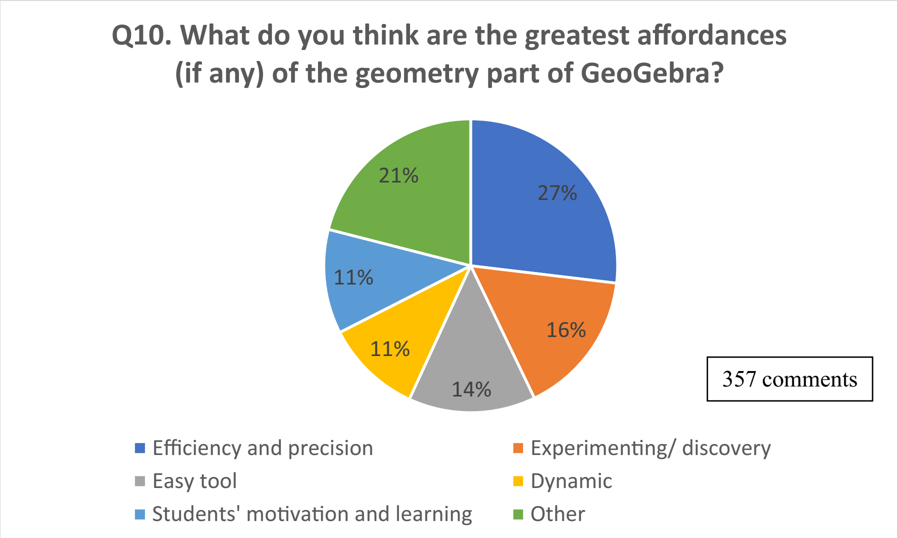

Fig. 1 Categories of comments to Q10

Another large category, 'Easy tool' (14%), contains comments that refer to the accessibility of the software. Examples of such comments are:

(iv) **Easy to use**.
(v) **Intuitive user interface**. *Large supply of tools.* **Straightforward guidance on the screen, when a tool is selected**.

A common feature of the comments in these two categories ('Efficiency and precision' and 'Easy tool') is the reference to the pragmatic value of the techniques, which are performed with the tool, presumably in comparison with the pragmatic value of the techniques performed in a paper-and-pencil environment, in relation to the same mathematical object. It may indicate a lack of respondent reflection on the epistemic value of the techniques or perhaps it reflects that a DGE is used to solve paper-and-pencil tasks, and therefore the difference in epistemic value of the techniques is not experienced eminently by the respondents.

The comments in these two categories do not directly refer to the four potentials. It may be argued that the reference to precision is related to the feedback potential, but the comments in these categories do not indicate that the feedback function is related to reasoning about constructions or making conjectures, but rather to alleviate poor construction skills in the paper-and-pencil medium. The finding supports the indication that many Danish lower secondary school teachers use GeoGebra as a substitute for the paper-and-pencil medium.

The category 'Experimenting/discovery' (16%) contains comments which are, as might be expected, characterized by reference to the possibility either of experimenting or of discovering. The comments typically do not state what the purpose of experimenting is, or what the experiments might yield, as exemplified in comment examples (vi) and (vii).

(vi) *It is an easy tool.* **Good for investigating**. *Draws nicely. Good for measuring.*
(vii) *It is fast and* **easy to investigate and try things** *and there are many tools that make the work easier for the students.*

Some of the comments do state the purpose of the experimenting, indicating discovery of relationships or truth of geometrical properties.

(viii) *That your work is precise.* **That you can conclude relationships on the basis of investigations**.
(ix) *Easy and clear.* **The students can easily experiment with the tasks, e.g. the relationship between area and length of sides**.

The category 'Dynamic' (11%) contains comments which directly refer to the dynamicity of the software. In the analysis, a merger with the 'Experimenting/discovery' category was considered, but a great deal of the comments in the 'Dynamic' category merely express that it is dynamic, without further elaboration. Therefore, the categories were kept separate, even if they overlapped in some instances. Examples of comments from the Dynamic category are:

(x) **That it is dynamic**.
(xi) *Fast solutions.* **Dynamic**. *The students work all of the time with mathematical concepts through the tools (*e.g. *perpendicular bisector, angle bisector). Motivating. Promotes development – more knowledge all the time.*

Although many of the comments in the 'Dynamic' and 'Experimenting/discovery' categories do not describe what or how it is dynamic, and what is or how it is being investigated, it is reasonable to surmise that the comments refer to the *epistemic* value of the techniques that are carried out in DGEs, in comparison with the epistemic value of the techniques performed in the paper-and-pencil environment. The comments in these categories also indicate that dragging is utilized to some degree, even though it is not explicitly stated, because experimenting, discovery and comments on dynamicity are likely related to dragging activity. For the same reason, it may be conceived that measuring in combination with dragging is utilized to some degree. Hence, the results support the indication that some Danish lower secondary school teachers use some of the potentials of DGEs in relation to RC.

The 'Students' motivation and learning' category (11%) contains comments referring to students being motivated by using the software or that the software supports students' learning either in general or in support of weaker students who might have problems, e.g. with fine-motor skills. The category 'Other' (21%) includes a variety of different comments, including a good portion describing the possibilities for solving tasks by drawing to find the solution (constructing) and some regarding visual or geometrical support. The comments from these last two categories do not refer to the four potentials.

Methodological and Analytical Issues

As mentioned previously, the questions posed to the teachers aimed at investigating to what extent the four potentials related to RC in Højsted (2019a, 2019b) were utilized.

However, even if the four potentials were utilized, it cannot be assumed in itself that the teachers are using DGEs in a manner coherent with supporting students' development of RC. For example, even though the teachers may report positively to Q7, indicating that "*students work on measuring of figures combined with dragging in order to investigate how measures change*", it is not guaranteed that elements of RC are a predominant learning goal of the students' activity. A positive report from the teachers in this case merely means that the potentials, which may be related to RC, are utilized to some degree.

Another relevant issue is that teachers may interpret the questions posed in ways which were not intended. Errors of this kind may be expected in relation to phrases such as, "*the difference between free objects and non-free objects*" (Q3) and "*robust figures*" (Q4), even if accompanying explanations of the terms were provided. In addition, teachers may have interpreted Q3 to concern specific free and non-free objects tasks, which they might work rarely on, or that they highlight this difference when appropriate, which may be frequently.

Discussion

Keeping in mind the methodological and analytical issues of the study, the results indicate that the potentials of DGEs in relation to RC, in particular measuring and dragging, are to some degree utilized in Danish lower secondary school. The particular utilization of the feedback potential in non-constructible tasks (Q8) seems also to be present, although at a lower rate. It may be regarded as a positive result, especially since dragging is considered to be a key feature of DGEs, one that affords a visual representation of invariant geometrical phenomenon allowing for generalization, reasoning and conjecturing (e.g. Arzarello et al. 2002; Laborde 2002; Baccaglini & Mariotti, 2010; Edwards et al. 2014).

However, the results from Q1 give rise to further questions about the actual utilization of these potentials. The results from Q3 indicate that the understanding of locked and free objects is not a particular focal point. As mentioned previously, the locked and free objects are the manifestations of the theoretical properties of figures, which are mediated perceptually in DGEs during dragging, thereby potentially linking the spatio-graphical and theoretical levels (Arzarello et al. 2002). Perhaps the lack of focus on this basic understanding can be linked to the students remaining at the spatio-graphical level when dragging. If that is the case, the conjectures will not be anchored in the theoretical properties of the figures, but at the spatio-graphical level.

Additionally, it seems that the tracing command is rarely used (Q9), which implies that dragging with trace activated in order to highlight underlying invariant relationships, is not presently utilized. 27% of the comments from the teachers in Q10 indicate, implicitly, that dragging and measuring are utilized to some degree, since experimenting, discovery and comments on dynamicity are likely related to dragging and measuring activity. These comments are also found to highlight the epistemic value of the techniques, which may be performed in DGEs. However, 41% of the comments refer to the pragmatic value of the techniques, the cause of which might be that many teachers use DGEs to solve paper-and-pencil tasks. Therefore, a considerable difference in pragmatic value of the techniques may be experienced by the teachers, whilst the difference in epistemic value of the techniques may not be experienced significantly. 73% of the comments were found not to be related to the four potentials of DGE in relation to RC.

Importantly, the extreme population of this survey must be taken into account in the discussion of the results and in answering the research question. As mentioned, the respondents are not representative of the average Danish lower secondary school mathematics teacher population. In fact, many of the respondents are teachers who regularly use GeoGebra in their classroom and may even be considered as "super users". Analysing the results in this light, it may be concluded that *the four potentials are scarcely utilized* in Danish lower secondary school.

The results are in alignment with those from Ruthven et al. (2004), showing that teachers mainly refer to the added pace and productivity provided by technology. There were also teachers who highlight the possibility of experiment/discovering, similar to Bozkurt and Ruthven's (2017) case study of a recognised expert secondary school teacher. However, only 16% of the comments were of this type.

Further Research: Integration of Findings into Practice

In order to improve the utilization of the potentials in relation to RC, it is necessary to increase the availability of tasks that are actually made for utilizing DGE potentials. Even though some teachers may adapt paper-and-pencil tasks, it would be beneficial for them to have support in the form of guidelines. Several DG task quality models have been suggested (e.g. Trgalova et al. 2011; Trocki and Hollebrands 2018). It cannot be expected that teachers, without any guidance, will adapt paper-and-pencil tasks into somewhat specialized tasks that utilize the potentials of DGE in relation to RC, such as soft construction tasks that can be solved by using the maintaining dragging model (Baccaglini-Frank and Mariotti 2010).

Additionally, it is necessary to highlight the mathematical meaning of free and locked objects in instruction and tasks, in order to support the students in linking the spatio-graphical and theoretical levels. This is important, since awareness of the theoretical relationship between elements of a figure, which is mediated perceptually by DGEs as invariants during dragging, is a premise for investigating figures in conjecturing and reasoning tasks, tasks which are characteristic of RC.

The insights gained from this research are integrated into another on-going related project (Højsted 2018), in which the aim is to develop guidelines for the design of didactic sequences that utilize the potentials of DGEs in relation to students' development of the RC (initial guidelines reported in Højsted (2019a)). The survey results suggest that the didactic sequence should include initial instruction and tasks aimed at supporting the students in understanding the theoretical underpinnings of locked and free objects, so that they can interpret the theoretical aspects of figures, which decide how the figure reacts when elements of the figure are dragged. Implementing 'construction tasks', as coined by Mariotti (2012), may support this process. It is also clear that the sequence must take into account the likely lack of teacher and student knowledge regarding the trace command and tasks instigating the construction of non-constructible figs.

In Conclusion

The study shows that Danish lower secondary school mathematics teachers scarcely utilize the potentials of DGEs in relation to RC in their classrooms. The teacher's comments reveal that DGEs are mainly used for pragmatic reasons, in particular for efficiency and precision. Additionally, there are indications that DGEs are used as a substitute to solve tasks that were originally designed for paper and pencil. To support the teachers' integration of DGEs' potentials into praxis, guidelines are being developed for the design of didactic sequences that utilize the potentials of DGEs in relation to students' development of the RC, which I hope to be able to report on in future publications.

Compliance with Ethical Standards

Conflict of Interest The author states that there is no conflict of interest.

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Paper III

Analysing signs emerging from students' work on a designed dependency task in dynamic geometry

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Analysing signs emerging from students' work on a designed dependency task in dynamic geometry[1]

This article reports on the design and implementation of a didactic sequence in the frame of a design-based research study. The research hypothesis is that affordances of dynamic geometry may support awareness of logical relationship between geometrical objects. We elaborate on the task design principles used in the study and present analysis of four Danish grade 8 students (age 13-14) working in pairs on the very first task of the sequence. The Theory of Semiotic Mediation frames the design of the study and the analysis of data, which was collected in the form of screencast, video and written products. The results indicate that students expect dependencies to be non-hierarchical in DGE; that low achieving students describing the behaviour of constructions during dragging refer to their global appearance; that specific prompts in the task design can shift students attention on specific elements of constructions; and that explicitly asking the students to explain any unexpected observation seems to be necessary for active reflection.

Keywords: dynamic geometry environments; task design principles; mathematical reasoning competency; design-based research; theory of semiotic mediation

1. Introduction

In the vast research literature on dynamic geometry environments (DGE hereafter), several studies deal with the relation between DGE affordances and students' mathematical reasoning, conjecturing and proof (e.g. Mariotti, 2012; Sinclair & Robutti, 2013; Hollebrands, Laborde & Sträßer, 2008). A seminal affordance of DGE, is that dynamic geometrical figures may be constructed, which may be manipulated by **dragging**, while certain properties remain invariant. The relationship between the

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elements of the figure is locked in a hierarchy of dependencies, determining the outcome of a dragging action (Leung, Baccaglini-Frank & Mariotti, 2013). These dependencies are linked to the theoretical properties of the figure, which are decided by the construction method, by the theory of Euclidean Geometry governing the system and by software design choices. Although the research literature on DGE affordances is comprehensive, Sinclair et al. (2016) state that task design and teacher practice remain understudied, a statement, which was reiterated by Komatsu and Jones (2018). An influential contribution in this domain is the Theory of Semiotic Mediation (TSM hereafter) (Bartolini-Bussi & Mariotti, 2008), which provides a framework for modelling the didactical integration of artefacts into school practice. The TSM provides a general frame for designing school activities aiming at exploiting the didactical potentialities offered by artefacts, in our case DGE, in relation to students' development of mathematical meanings, and the role of the teacher in this regard.

In this article, we report on the experimentation of a didactic sequence, which was carried out in lower secondary school in Denmark, focusing especially on the phase of task design. The aim of the entire didactic sequence concerns reasoning, conjecturing and proving. However, the results reported in this article concern the initial part of the sequence, in which the focus is on the role of the popular DGE software, GeoGebra, at fostering student awareness of theoretical properties of figures. Specifically, we focus on the design of **dependency tasks**, which is tasks aimed at fostering students' awareness of dependency properties in a GeoGebra figure (a detailed description is provided in section 4).

In the next section, we briefly introduce key concepts from the TSM, after which the research question of this study is formulated. Afterwards, the method of the study is presented, followed by a description of the objectives, hypotheses and choices concerning the task design of the initial tasks in the didactic sequence. Then we present the analysis of data from two pairs of students working on the very first task in the sequence. Finally, we discuss the results in light of the research question.

2. Conceptual framework

The TSM (Bartolini-Bussi & Mariotti, 2008) characterizes how students form meanings in relation to the use of an artefact to solve a task. The initial **personal meanings** developed by the students as they use the artefact to accomplish a specific task, may not match the **mathematical meanings** an expert mathematician (the teacher) would recognize. However, through a didactical intervention, the evolution into mathematical meanings may occur. Bartolini-Bussi and Mariotti (2008, p. 754) coined the term *semiotic potential* of an artefact to express the duality of possible personal meanings and mathematical meanings, which may be evoked by using an artefact in the solution of a specific task. Awareness of this potential enables the teacher (or researcher) to design tasks aiming to promote certain mathematical learning. As the students work on the tasks, they will produce signs in the form of verbal utterances, written products, gestures and on screen DGE interactions. Underlying these signs are personal meanings of the students. The meanings may, to a varying degree, be coherent with the mathematical meanings laid out as the objective of the task. The expert teacher may act as a mediator that supports the students' evolution of mathematical meaning in relation to the artefact activity. The mediation may take place as the teacher interacts with the students, in particular in classroom discussions. When the artefact is intentionally used by the teacher who is aware of her/his pivotal role in managing this process, then it becomes a *tool of semiotic mediation* (Bartolini-Bussi & Mariotti, 2008, p. 754).

In the TSM, cognitive development is described from a Vygotskian (1978/1930) point of view, as a process of internalization, which has "*two main aspects: it is* *essentially social; it is directed by semiotic processes. In fact, as a consequence of its social nature, external process has a communication dimension involving production and interpretation of signs.*" (Bartolini-Bussi & Mariotti, 2008, p. 750). Therefore, the analysis of the internalization process may be oriented towards the analysis of the use of signs in social activities. In other words, the evolution of student meanings may be analysed by interpreting the signs the students produce, e.g., gestures, verbal utterances, written signs or DGE actions in social activities.

The analysis may unveil to what extent the specific designed tasks can function as expected, fostering the unfolding of the semiotic potential of the artefact. Testing the effectiveness of task design consists in controlling if the students show the expected behaviour, referring to the expected personal meanings described in the a priori analysis (presented in section 4) that are consistent with the mathematical meaning that is the objective of the didactical intervention. The development of meanings can be highlighted by identifying specific semiotic chains, e.g. chains of relations of signification (Bartolini-Bussi & Mariotti, 2008, p. 756).

As mentioned, the teacher plays an essential role in supporting the evolution of personal meanings toward mathematical meanings. However, interpreting and reacting in classroom discussions, sometimes on the spot, to signs produced by the students may be challenging for the teacher. Therefore, it may prove useful to accompany the design of the tasks both with the analysis of the semiotic potential and the description of the possible 'unfolding' of such a potential. In this way, the teachers can be equipped with guidelines concerning the type of signs, which can be expected to emerge as the students work on specific artefact tasks. In this light, the following research questions arise: *As students work on a* ***designed dependency task****, which type of signs emerge that are related to the use of the dragging tool and can be seen as evidence of students'* *awareness of the logical relationship between the geometrical properties in play? How can the unfolding of the semiotic potential from this case contribute to the formulation of guidelines?*

3. Method and context

The study is part of a larger project that is methodologically anchored in design-based research (Bakker & van Eerde, 2015), which is characterized by its dual purpose of developing educational practice as well as theory about the domain specific teaching and learning of that practice. Design based research is therefore *"claimed to have the potential to bridge the gap between educational practice and theory"* (Bakker & van Eerde, 2015, p. 2). The aim of the project is to develop guidelines for designing didactic sequences that utilize the potentials of DGE in relation to students' development of mathematical reasoning competency[2] (Højsted, 2020a; 2018). Based on an initial theoretical analysis, a 15 lesson didactic design was developed and tested in three design iterations in three different Danish 8th grade classes (age 13-14). In this article, we present results from the second iteration.

In our data collection, the students worked in pairs using GeoGebra on one computer to solve the tasks from a worksheet. The organization into pairs was chosen to foster interaction and communication in alignment with the request of production of signs that is a part of the TSM. In addition, from a methodological point of view, the organization into pairs allowed us as researchers to identify and analyse students' personal signs emerging from the work with the artefact when they interact and

[2] The mathematical reasoning competency is one of eight mathematical competencies in the Danish KOM framework (Niss & Højgaard, 2019)

communicate. The pairing of students was also a consequence of task design choices in order to meet educational aims, e.g. requiring the students to explain and justify to each other, which is a characteristic of reasoning competency (we will elaborate further in section 4).

Data was gathered in the form of screencast recordings from all groups and collection of students' written products. In addition, three groups were chosen in each class for external video recording to allow for a richer collection of emerging signs. The groups were chosen in collaboration with the teacher to comprise a range of low to high attaining groups concerning mathematics achievement.

The data is analysed by identifying the emerging signs/semiotic chains of the students, in order to make a synthesis of possible personal signs that the teacher may expect, and to review to what extent the meanings are aligned with the expected outcome of the task design. Finally, the design is evaluated in light of the analysis and some refinements of the design are proposed. In this article, we present data from two groups, one medium-high achieving group and from one low achieving group. We do, however, also refer to data from other groups in the analysis and concluding discussion.

DGE are commonly used in Danish primary and lower secondary school (GeoGebra, in particular, is popular), and it is explicitly mentioned in the mathematics curriculum, e.g. describing that students should be able to draw/construct figures using a DGE at the end of grade 3 (Børne- og Undervisningsministeriet, 2019). Although DGE is common, there are indications that GeoGebra is mainly used as a substitute for the paper and pencil environment, and teachers highlight pragmatic means, in particular the efficiency and precision that the software offers (Højsted, 2020).

The students in this study had basic previous GeoGebra experience, which means that they knew the layout of the program, i.e. where the commands for construction are situated, and how to use some of these commands.

4. Task design principles

The design of the initial tasks in the didactic sequence can be decomposed into three related dimensions. At the macro level, there is an *objective,* which describes the students' learning aim. Then there is a *hypothesis* about the types of tasks, which may support the students to achieve the aim. Finally, there are *choices* made in the micro level of design, such as formulations in the task and descriptions of requested student activity. To ensure alignment, each choice should be coherent with the hypothesis, which in turn should be coherent with the objective. This structure is methodologically consistent with the predictive and advisory nature of design-based research (Bakker & van Eerde, 2015) and offers a systematic and explicit support for linking the design process to the revision.

The learning **objectives** are twofold: (1) that the students develop an awareness of the logical dependency between geometrical properties of dynamic figures[3] in GeoGebra. That involves being able to discern free and locked objects in GeoGebra, and to be aware of the fact that it is these relations between objects, which decide the

[3] Such dynamic figures are complex entities that represent geometrical objects (denoted "Cabri-figures" by Laborde and Laborde (1995)), like others, they are images but they are images on the screen and the product of a digital construction. Any construction determines a logical relationship between geometrical properties of such dynamic figures, which are perceptually observable as invariants during dragging.

outcome of dragging. (2) That they are able to interpret the construction dependency geometrically as logical dependency. This requires geometrical attention to the theoretical relations induced in the construction procedure.

The **hypotheses** concern both these objectives and are related to the semiotic potential of a DGE with respect to the logical dependency between geometrical properties of a constructed figure (Leung, Baccaglini-Frank & Mariotti, 2013; Mariotti, 2014). The semiotic potential already described in relation to a construction task, is now reformulated from the perspective of the task design of a "dependency task" in terms of objective, hypotheses and choices. The hypotheses are based on the previous literature concerning the semiotic potential of tools in a DGE. Hypothesis (1): Since any constructed figure behaves according to the geometrical relationships defined by its construction procedure, students acting on a figure produced by a construction command can observe the invariance of a property or the invariance of a relationship between properties (Mariotti, 2014), and the perceived invariants can be related to the construction process.

Hypothesis (2) concerns the semiotic mediation process. The students' perception of the phenomena observable on the computer screen may be linked to a geometrical interpretation. Partly, this geometrical interpretation may occur spontaneously if the students utilize their previous geometrical knowledge, but in particular, such interpretation can be fostered by specific semiotic activities (discussing tasks in pairs, explaining and writing the description of what they observe), and most essentially, through the mediation of the teacher in classroom discussion. Even though hypothesis 2 is of utmost importance from a teaching/learning point of view, we will primarily focus on hypothesis 1 in this article.

The general choice of proposing the exploration of a constructed figure, is elaborated in the following five **choices** that are made at the micro level of the task design, in alignment with the hypotheses. (i) *A construction is proposed that contains certain dependencies, leading to a clearly recognizable invariant.* This choice is related to our aim of exploiting the semiotic potential of dragging in DGE to reveal invariance of properties. (ii) *The students are requested to produce the construction with guidance.* The choice reflects that the goal is to foster awareness of properties in the construction. Therefore, the students perform the commands themselves and in so doing are expected to reflect on which properties they are supposed to induce in the construction. In addition, they may interpret the behaviour of the construction after dragging, as a consequence of their construction method, and are therefore expected to reflect on the possible consequences of the construction steps. Some guidance was given in the form of accompanying pictures of commands, which may be useful to complete the construction, as well as a picture of the required construction (see example in figure 1). Choices (iii-v) are related to what White and Gunstone (2014, p. 44-65) refer to as Prediction-Observation-Explanation, and concern the selected types of requests for the students: The students are required to predict the result of an event, and to justify their prediction. Afterwards, they are required to report what they observe and explain their observation, resolving any differences between prediction and observation. (iii) *The students are encouraged to predict, before they drag objects, what will happen on the screen when they drag certain points, and to justify their prediction to the co-student they are working with.* Asking the students to predict the properties of the diagram before they drag, directs their reflections onto properties of the construction (the general objective) and may give rise to conflict, if what they observe does not coincide with their prediction. The conflict can provoke intellectual curiosity (Laborde, 2003).

Encouraging students to justify their prediction serves two aims; firstly, it supports the development of a mathematical attitude to look for a reason - to justify the conjecture. It can be named a social mathematical norm (Yackel & Cobb, 1996): when I formulate a statement, a reason should be provided. Secondly, it supports the production of reasons that can develop into mathematical reasons. Both these types of support concur with the aim of the students becoming able to justify mathematical claims to others, which is a characteristic of the mathematical reasoning competency (Niss & Højgaard, 2019, p. 16). (iv) *The students are encouraged to drag certain points and to describe what happens.* This choice is added so that the students can confirm the expected outcome, or wonder why it did not go as expected and try to figure out why. Again, with the goal of students becoming aware of the relationship between the properties induced by the construction and the properties that appear invariant by dragging. (v) *The students are encouraged to give an explanation concerning certain essential relations in the construction.* This choice may direct the students' attention to certain essential properties of the construction, again to pursue the main goal of developing awareness of the theoretical properties of the constructions, which decide the outcome of dragging. Choices (iii-v) are sometimes repeated for different elements of the same construction.

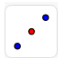
We denote this type of task, which encourages the construction of a figure and consequent guided exploration of the dependencies in the figure, a **"dependency task"**.

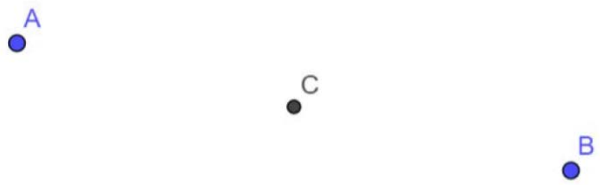
According to the TSM, the request of discussing and writing that accompanies each task constitutes the semiotic component of the design related to hypothesis (2); it is expected to trigger the semiotic mediation process that is rooted in the use of the artefact. What emerged from these semiotic activities will constitute the base on which the teacher is expected to plan the plot of the following classroom discussion. This material also constitutes the corpus of data, which the analysis of the unfolding of the semiotic potential is based on. The specific choices concerning the role of the teacher in the classroom discussion will not be elaborated upon in this article.

4.1. Task 1 – a dependency task

In the data presented in this article, the students are working on the very first task of the sequence, which was a dependency task. Below we present the task formulation that is related to choices (i-iv)[4]:

"1.a. Construct two points A and B in GeoGebra and the midpoint C between them.

Use the midpoint command 



1.b. What do you think happens to the other points when you drag point A? Guess first and justify your guess to your partner.

Investigate afterwards, what happens?

[…]

1.d. What do you think happens with the other points when you drag point C? Guess and justify first.

Investigate afterwards, what happens?"

Table 1. Task 1 – a dependency task.

[4] We omit the formulation of 1.c., 1.e. and 1.f. since it is not discussed in this article. 1.c. was a repetition of 1.b. in relation to point B, while 1.e. and 1.f. concern choice (v).

5. The unfolding of the semiotic potential – presentation and ensuing analysis of students' signs emerging from the activity

The organization into pairs and the requests (Predict-Observe-Explain) are expected to foster interaction and specifically communication. Therefore, the requests are expected to foster the emergence of signs and underlying these signs are personal meanings of the students.

In the following, episodes in the form of transcripts, screen recordings, video and written work of students are analysed to interpret underlying meanings. Since we are interested in the unfolding of the semiotic potential, the analysis will focus on the production of signs related to the use of the artefact, and related to the idea of logical dependency between geometrical properties. We seek to identify emerging personal signs of the students in order to make a synthesis of possible personal signs that the teacher may expect, and thereby be preparing the teacher to support the evolution of mathematical meanings in the classroom discussion. In addition, the choices made in the task design are evaluated in light of the analysis, and refinements of the task design are proposed.

5.1 The case of Dan and Jan

The teacher described Dan and Jan as average achievers in the mathematics classroom. Unfortunately, the guessing part of their work on task 1.b was not recorded, but from the written work we can see that Dan and Jan have guessed "when you drag A, they will move along". It seems that they think that both C and B will move.

We now look at Jan and Dan's emerging signs as they are investigating what happens (choice (iv)). When sequences of pictures are shown, the picture on the left is first in the chain.

8 Jan: Ok then we're trying to move it. *(drags point A (Figure 1))*

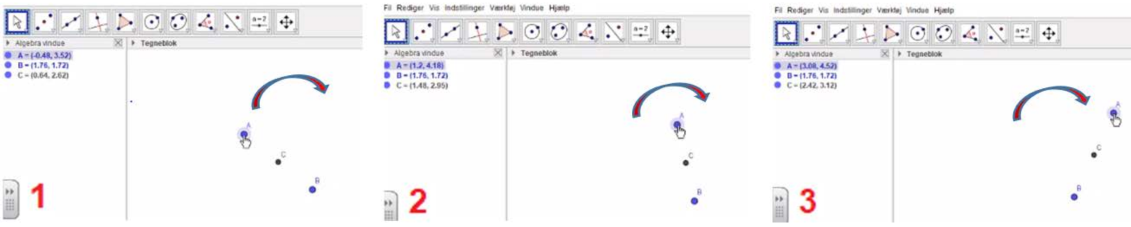


Figure 1. Screencast recording of Jan dragging point A with arrows added to specify dragging direction.

9 Jan: They move along in parallel. *(gestures with his hand) [indicating that the three points A,B, C (or perhaps only A and C) move while remaining on a line, or moving on a trajectory that is related (Figure 2)]*

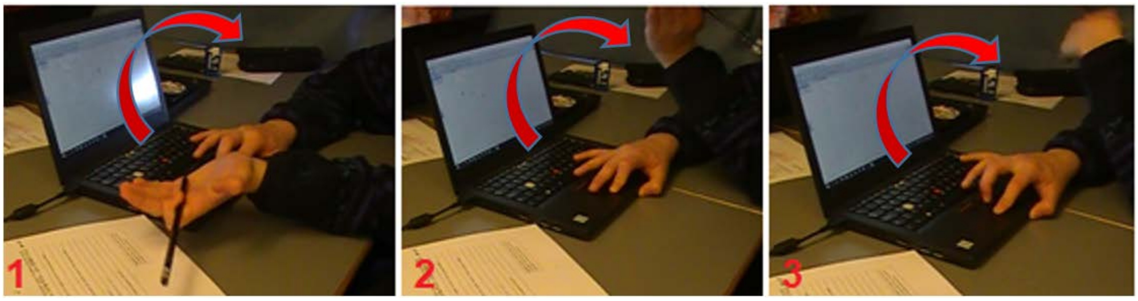


Figure 2. Video recording of Jan gesturing with his hand. Arrows are added to illustrate the direction of the movement.

10 Dan: They move along in parallel, do you think we should write it? *(writes down 'parallel' on the line where the guess was written)*

12 Dan: Move along parallel, good.

[…]

17 Jan: Just write they move along with them… Or what do you call it? *(gestures with hand) [indicating a line (Figure 3)]*

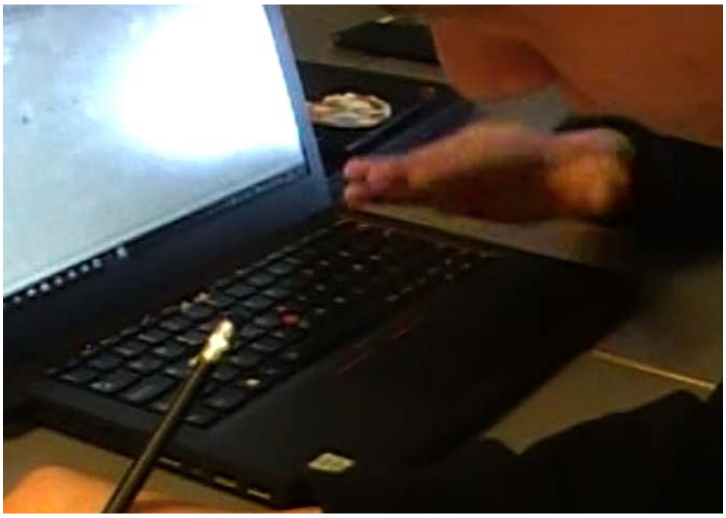
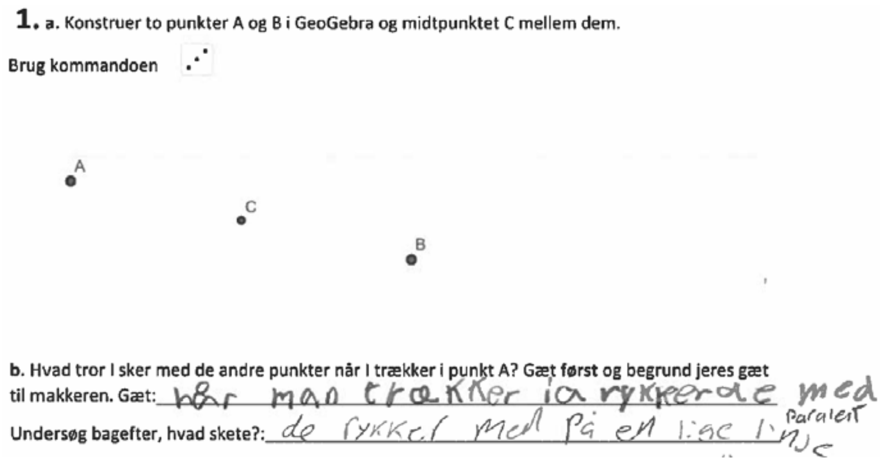


Figure 3. Jan gesturing with his hand to indicate a line.

18 Jan: they are moving in a straight line.

19 Dan: *(writes and talks)* Move along on a straight line

Their written answers are shown in figure 4:



1. a. Konstruer to punkter A og B i GeoGebra og midtpunktet C mellem dem.

Brug kommandoen

b. Hvad tror I sker med de andre punkter når I trækker i punkt A? Gæt først og begrund jeres gæt til makkeren. Gæt: når man trækker i a rykkerde med paralelt

Undersøg bagefter, hvad skete?: de rykker med på en lige linje

Figure 4. Dan and Jan’s written answers to question 1.b.

Translated: Pre investigation answer “When you drag A, they will move along in parallel” and post investigation answer “They move along on a straight line”.

Note that “in parallel” was in fact added to their prediction answer after the investigation, meaning that they revised their guess based on the investigation.

5.1.1. Analysing task 1.b.

If we take a closer look at the semiotic bundle[5] (Arzarello, Paola, Robutti & Sabena, 2009; Arzarello, 2006) consisting of the verbal utterance from line 9. “They move along in parallel” and the accompanying gesture (movement of Jan’s hand), we may, from a semiotic perspective, interpret the underlying personal meaning being that the points are moving while remaining on an invisible segment, although this is not expressed fully yet. The unexpected usage of the notion parallel is an interesting geometrical interpretation given by the students. It is not appropriate from a mathematical point of view; however, the movement perceived must somehow remind them of the notion of parallelism. We can make the hypothesis that they use this sign, ‘parallel’, consistently with its use in the common language. Acatully, such an expression – move along in parallel - is commonly used in Danish to indicate that something is happening at the same time. The use of this expression in order to describe what is observed is also compatible with some figural aspects of the geometrical meaning that these students may have for these expressions; for instance, parallel lines may be evoked by the movement of the points.

In lines 17-18 Jan refines his own expression, “move along in parallel”, into an expression that somewhat more accurately describes what happens, they move along on

[5] Arzarello (2006) introduced and elaborated (Arzarello, Paola, Robutti & Sabena, 2009) the terms semiotic set and semiotic bundle. A semiotic set comprises three components: signs; modes of production/transformation of signs; and relationships among signs. A semiotic bundle is a collection of semiotic sets and a set of relationships between the semiotic sets of the bundle (Arzarello, 2006, p. 281).

a straight line. This strengthens the indication that “moving along parallel” meant moving along on an invisible segment.

By saying “**they** move”, they seemingly indicate that point B moves, which it does not. Perhaps the students are preoccupied with the *global change of appearance* of the construction and once they have identified the invisible line connecting the points, then, in their view, point B follows the movement of the other points, as a part of the line, in a rotation about itself. However, it is also possible that “they move” refers only to points A and C, and that the students are expressing that if A is moved approximately along a straight line trajectory, the midpoint C will move also on a linear trajectory, and this will be parallel to the trajectory of point A.

It is noteworthy that the students do not describe the invariant AC=BC, but that they do describe that the points remain on an invisible segment. Perhaps it is less immediate to notice covariant lengths than noticing mutual positions. It is consistent with the gestalt rules of perception, in which perception of a form is conceived as a global structure (Wertheimer, 1958). In our specific case, the gestalt rule of *similarity* seems to guide the students’ organisation. The rule asserts that there is a tendency to see a form so that similar aspects are grouped. Dan and Jan seem to “see” the similarity of the movement of points A and C (and perhaps B), remaining on a trajectory that is related.

If we analyze the signs from Dan and Jan in relation to the aim of choice (iv) *The students are encouraged to drag certain points and to describe what happens*, we may surmise that the outcome of dragging was not quite what they expected, because they try to reformulate what happens based on the investigation. They proclaim, “they move along in parallel”, which is then further refined into “they move along in a straight line”. However, the students do not explicitly connect what happens during dragging with the theoretical properties induced by the construction, neither in the first guessing nor after the investigation. No explication of the movement that they observe is given referring to the construction process or to the geometrical properties induced by such construction. Although the students do indeed drag and describe what happens, they do not guess or “see” what was aimed for. This might be explained by formulation of the task. The students are asked to guess and investigate “what happens to the other points”. The formulation does not refer to a possible geometrical interpretation of what can be observed. No explicit question is posed about the interpretation of the phenomenon that can be observed. There are many possible answers to the question “what happens”. It does not necessarily mean that choice (iii-iv) are invalid in terms of accomplishing the hypotheses, but rather that the realization of the choices may need to be refined. The formulation of the request of observation is too generic, consequently the answer is generically controlled by the basic rules of perception and a *perceptual apprehension* is privileged instead of a more *discursive apprehension* (Duval, 1995) that could orient the observation on the two segments AC and BC, into which the original segments AB is divided, and on their congruence.

5.1.2. Task 1.d data.

As shown previously, task 1.d. is also related to choices (iii) and (iv), but now it is point C that is to be dragged.

We now return to Dan and Jan as they are about to make a guess. The screen at that moment is captured in figure 5.

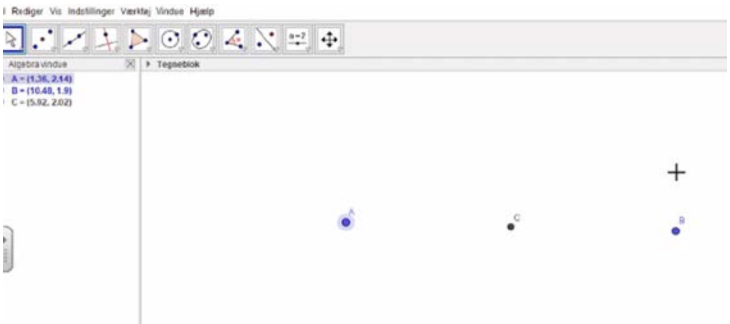


Figure 5. Screencast recording as Dan and Jan make their guess to task 1.d.

33	Jan:	I think that when you move point C, then they move *[probably means A and B]*
34	Jan:	just like *(moving his hand up and down as if it was a line (Figure 6))*

Figure 6. Jan moving his hand up and down.

35	Jan:	so if we move it here, and then drag it *[talking about point C]*, then they move like this *(moving hand up and down again)*
36	Dan:	move along in parallel.
37	Jan:	hm, it is called parallel?... *(moves his hand i horizontal direction (Figure 7))*

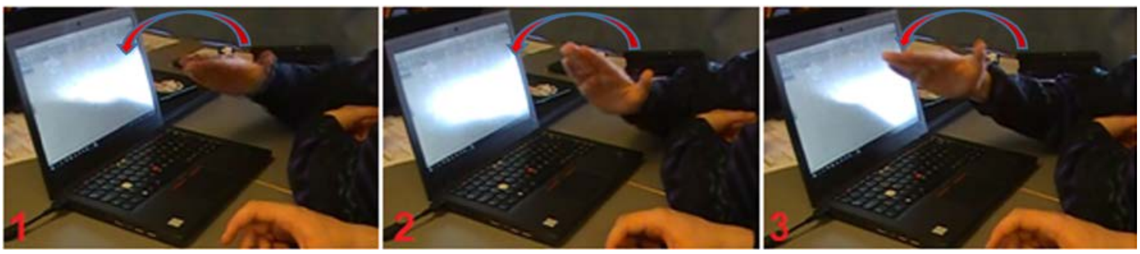


Figure 7. Jan moves his hand in a horizontal direction.

38	Jan:	Yes parallel to
39	Dan:	Horizontally
40	Jan:	Like if you have a stick *(Figure 8)*

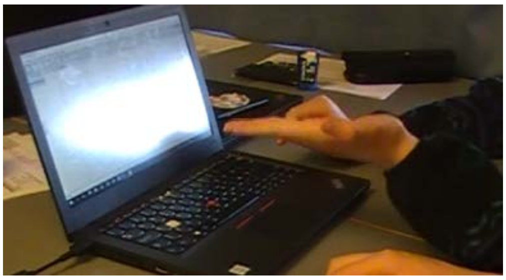


Figure 8. Jan uses his hand to represent a stick

41 Jan: and lift it in the middle *(Figure 9)*

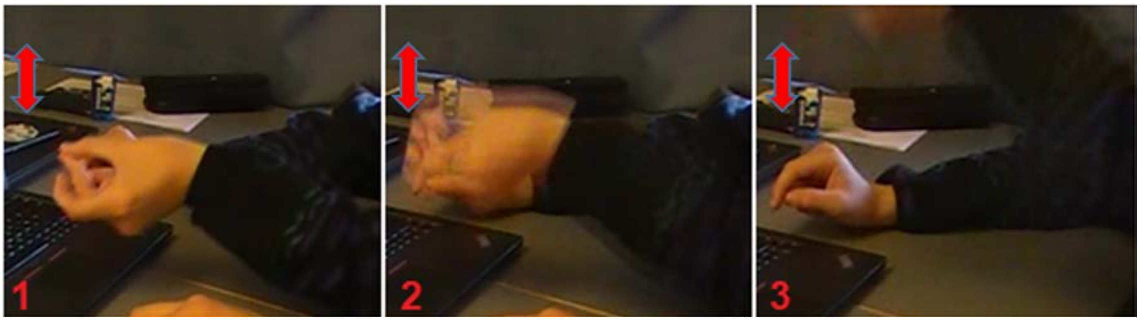
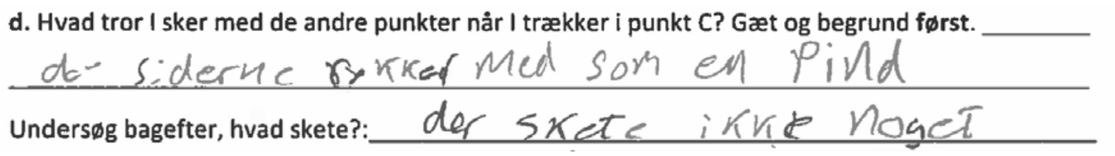


Figure 9. Jan gestures holding the imaginary stick by its centre and moves it up and down.

42 Dan: and in parallel
43 Dan: The sides *[the extreme of the stick]* move along
44 Jan: Exactly
45 Dan: *[writing and talking]* The sides move along
[…]
48 Jan: Let's try *(Tries to drag C)*
49 Jan: Nothing is happening... It should.
50 Dan: Nothing happens, huh.
(They are puzzled)
51 Jan: No…
(Dan writes: nothing happened)
52 Jan: *(asks some other students)* Did something happen for you with C? *(does not get a response)*
53 Dan: Nothing happened
54 Jan: You are not supposed to use this one? *(clicks on Move graphics command),*
55 Dan: no, that one *(indicating the move command)*
56 Jan: …What's next? *(they move on to the next task)*

Written answers in figure 10:


d. Hvad tror I sker med de andre punkter når I trækker i punkt C? Gæt og begrund først. de siderne rykker med som en pind

Undersøg bagefter, hvad skete?: der skete ikke noget

Figure 10. Dan and Jan’s written answer to question 1.d.

Translated: Pre investigation answer, “the sides move along like a stick” and post investigation answer “nothing happened”.

5.1.3. Analysing 1.d.

In lines 33-35, it seems that Jan is finding it hard to express in words what he expects, but his gesture suggests that he expects the invisible segment to be maintained, meaning that A and B will follow. Again, the students use “move along in parallel” (lines 36-38). However, Jan tries to express in more detail, or to refine the expression, that the extremes (A and B) will move along, as if they are extremes of a stick, which is gripped at its centre and raised (lines 40-41). They are puzzled when nothing happens (line 49-51), which makes Jan enquire with other students and question whether or not they are using the correct command. However, they spend less than a minute on trying to find out why it was not possible to drag point C, before they move on to the next task without being able to find a good reason.

If we interpret the meaning underlying these signs, it seems that the students, in particular Jan, is aware of the fact that there are relations between the points. Perhaps this awareness is not at the level of points, but certainly on the global level of the construction. It seems that the elements of the construction are considered integral one to each other and hence the construction is as a “rigid system”, to borrow a notion from physics. While this expression “stick” to talk about the configuration indicates that the students are aware of constrained relations induced by the construction, it also unveils that students do not intuitively expect that there should be a hierarchy of dependencies.

They envisage that the construction behaves as a stick would, i.e. if you lift a stick by its middle, the sides will follow, and vice versa. When they are puzzled, they look for an explanation in the technology. It seems that there is no awareness of the relationship between the functioning of the software and the Geometry that is embedded. However, the surprise does seem to awaken an interest. This may lead us to the hypothesis that a good choice would be to make the students observe a situation where they expect something to move together and on the contrary, it does not.

In the view of the aim of choice (iii), it is evident that Jan tries to predict what will happen (lines 33-41), and the guess seems to some degree to be embedded in reflections on the relations stated in the constructions, however, it does not look as if he intends to justify his guess to Dan. It may be argued that his description of the construction moving like a stick, is in itself a justification of the movement, namely that the construction is “rigid”. Nevertheless, the justification is vague and the reflections remain at a global level of the construction, i.e. there is no mention of points or relations between points. If we consider the aim of choice (iv), then the students do indeed describe what happens, however, they make almost no effort to figure out why it is not possible to drag point C, and mainly they do not refer to properties induced by the construction.

5.2. The case of Sif and Ole

In this section, we present and analyse part of the data from two other students from the same classroom, Sif and Ole, also working on task 1. The teacher described Sif as a high achiever and Ole as an above average achiever in the mathematics classroom. They are about to make a guess to question 1.b:

18	Ole:	Ehm, I think B stays in the same place and A will be moved and C will still be in the middle. [*Pointing to the points onscreen with his finger while explaining*]
19	Sif:	eh, B stays in the same place. And then
20	Ole:	C remains in the middle
21	Sif:	[*writing the answer and talking*] C will move, depending on A's... will move to stay in the middle.
22	Sif:	[*Reading the text out loud*] Investigate afterwards what happened. [*Drags point A*]*(Figure 11)*

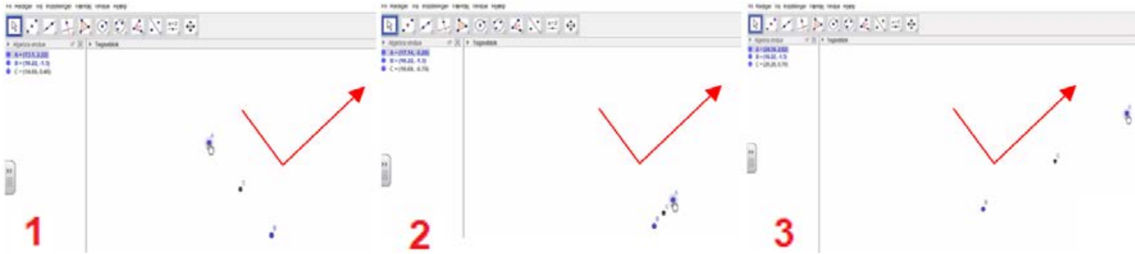


Figure 11. Sif drags point A.

26	Sif	[*writing and talking*] We guessed right

Their written answers are shown in figure 12:

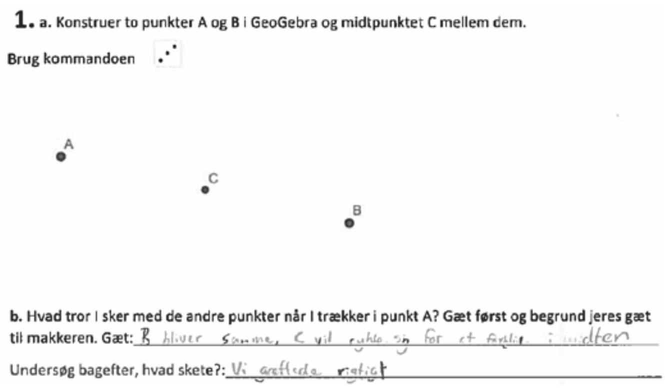


Figure 12: Sif and Ole's written product.

Translated: Pre investigation answer "B stays the same, C will move in order to remain in the middle" and the post investigation answer "we guessed right".

5.2.1. Analysing 1.b.

Sif and Ole describe the expected movement of each point in the construction. Their expressions indicate an awareness of the fact that the relations induced by the construction will be maintained. The description is at the local level of points involved in the construction: “B stays in the same place”, “A will be moved”, “C will still be in the middle” (line 18). We can also notice that the utterance about point C is elaborated upon to highlight that it will not stay in the same position but rather that “C remains in the middle” (line 20). Further, they indicate that C is dependent on A, “C will move, depending on A’s…” (line 21). We may interpret that the personal meanings underlying the expressed words are coherent with the meanings that the activity aims to foster, namely that there are relations between the geometrical objects, that these relations determine the dependency between the points and such relations decide the outcome of a dragging action. The students do in fact explicitly express such a dependency as the final result of a semiotic chain that evolves in the dialog between the two students. We can see how the meaning of dependence becomes more and more explicit in the semiotic chain: “still be”, “remains”, “move depending on A and move to stay in the middle”. In the last formulation – that is not reported in the written report – both the dependence relation and the specific property originating the dependence are made explicit.

Considering the aim of choice (iii), the students do make a prediction based on the theoretical properties, which they induced in the construction. Sif also offers an explanation to the expected movement of point C, “C will move, depending on A's... will move to stay in the middle.”, but not for the other points. The students drag and confirm their prediction, which was the aim of choice (iv).

5.2.2. Task 1.d. data

The following occurred as Sif and Ole worked on task 1d:

95	Sif:	[*reading the text*] What do you think happens to the other points when you drag point C? Guess and justify first.
96	Ole:	It's all moving together.

97	Sif:	Then everything moves because C must be in the middle. Then they will move in relation to C? [*the tone indicates a question and she looks at Ole*]
98	Ole:	I think so.
99	Sif:	Then one could imagine that it was just a line moving around *(figure 13)*

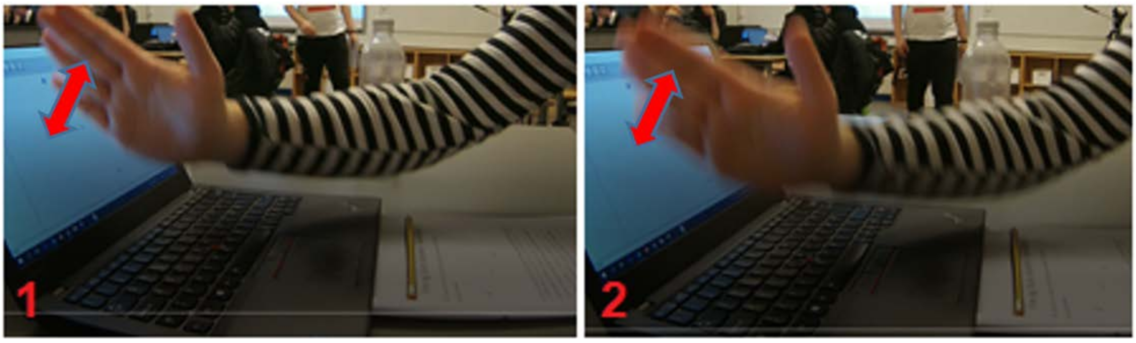


Figure 13. Sif gestures with her hand to represent a line going through the three points. It looks like she moves her hand so that it remains parallel to the initial line.

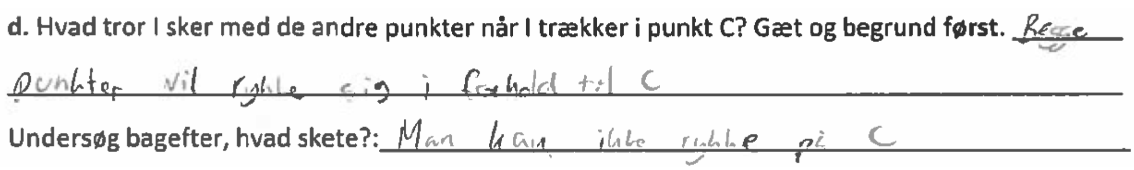
100	Ole:	Yes exactly, in parallel.
101	Sif:	Okay, so we just say that everything will move relative to point C. [*writing*]
103	Sif:	Yes, because it must be in the middle in relation to C. [*Sif tries to drag point C*]
104	Ole:	Oh!
105	Sif:	One cannot move C. [*Sif writes down*]
109	Sif:	Ehhh, and why can't you? ...

[*the teacher has stood next to them for a while, and decides to intervene*]

110	Teacher:	Why can't you move C?
111	Ole:	Eh, I don't know
112	Teacher:	Why can't you move the midpoint?
113	Sif:	It is perhaps because it is the midpoint in relation to the other two points.
114	Teacher:	What do you think you can move if it was? [*it seems he is asking them what is possible to move*]
116	Sif:	A and B.
117	Teacher:	Yes you can move on A and B because that was what he said [*"he" means the researcher, which did an introduction to the material at the start of the lesson*], you know, it's a dynamic program. That is, C will always be the midpoint, that is, C is automatically moved if A and B are moved.
118	Ole:	Yes exactly.

119 Teacher: If you do not move A and B, then C stays the same.
120 Sif: Okay. [*they move on to the next task*]

Written answers are shown in figure 14:


d. **Hvad tror I sker med de andre punkter når I trækker i punkt C? Gæt og begrund først.** Begge punkter vil rykke sig i forhold til C

Undersøg bagefter, hvad skete?: Man kan ikke rykke på C

Figure 14. Sif and Ole’s written answer to question 1.d.

Translated: Pre investigation answer “Both points will move in relation to C” and post investigation answer “You cannot drag C”.

5.2.3. Analyzing 1.d.

We may interpret from line 96 “It's all moving together” that Ole intuitively expects that the construction will move as a rigid structure. This immediate expectation is similar to Dan and Jan’s “stick”, and can be observed in other groups as well. Sif understands Ole’s suggestion, elaborates upon it and justifies why it may be so, based on the construction. However, she is not completely sure (line 97). Although she says that the line moves “around”, her gesture suggests a movement remaining parallel with the initial position in an orthogonal translation (line 99). In this case, the semiotic bundle (speech +gesture) is used to express movement and the specific kind of movement. Because there is the need to express a particular kind of movement the combination between speech and gesture allows to express what otherwise, only verbally, would be very difficult to say. Ole notices the meaning of the gesture and makes it explicit (line 100).

The students predict what will happen and justify their predictions, based on theoretical properties, which was the aim of choice (iii). The description of the expected global movement of the construction is interwoven with justifications based on local elements such as “everything moves because C must be in the middle” (lines 97, 101, 103). The surprising result of their dragging investigation leads them to wonder why C cannot be moved. It seems plausible that they would continue to work on this question if the teacher did not intervene, hence choice (iii) worked according to plan in this case. The initial teacher action is promising: first, he asks “why can’t you move point C” followed by a reformulation into “why can’t you move the midpoint”. This reformulation is not neutral, it highlights the theoretical status of point C. The intervention of the teacher moves from a general to a more specific question. Such a shift leads the student to immediately grasp the suggested geometrical perspective and guess (line 113). However, the teacher then unfortunately shifts the focus from non-draggable to draggable points, to which the students correctly answer A and B. He explains what they already know, that C remains the midpoint, referring to an authority argument and to the software “that is what he said, you know, it’s at dynamic program” (line 117), and finally “If you do not move A and B, then C stays the same” (line 119). His explanation does not make it any clearer why C cannot be moved. What is observed can be explained both by geometrical reasons and by software design reasons. In GeoGebra, it is not possible to drag locked objects, which are derived from other objects by construction. However, other DGE[6] allow dragging the dependent points too. The teacher’s intervention, which was not requested by the students, seems to close the door on the students’ investigation, and can be considered a missed learning opportunity.

[6] E.g. Geometer Sketchpad 3, as mentioned in Talmon & Yerushalmy (2004)

6. Discussion

Referring back to our research question, and in relation to the results of the analysis presented above, we will discuss the type of signs that emerge as the students work on the designed dependency task in relation to the goals set out for the activity, and consider how the task design may be revised accordingly.

We observe the recurring sign “move along in parallel” in the cases above. The sign has a particular meaning to the students, and the meaning comes from the artefact as they try to describe what can be observed after dragging and elaborate on what they expect. Specifically, it emerges when the students experience situations involving covariance. The explanation concerning the use of the notion “parallel” may be found in the fact that it is used commonly in the Danish language to mean “at the same time”. Thus, the students are using this expression not in a geometrical sense, but in a common sense. They attempt to describe what appears on the screen, but do not interpret it geometrically, and therefore, the outcome is not consistent with our didactical objective of introducing students to interpret the perceptual experience geometrically.

Many students, including the students from the two cases presented, expect the construction to behave as a rigid structure during the dragging of derived points. This finding indicates that students do not immediately expect the relations between elements to be hierarchical that is based on interdependency. In fact, the students seem intuitively to view the construction as a rigid structure with non-hierarchical dependencies. Talmon and Yerushalmy (2004) observed related results in their study reporting on junior high students and graduate students in mathematics education. They found that the students predicted a dynamic behaviour that was “contrary to the behavior that would be expected based on the order of construction” (Talmon & Yerushalmy, 2004, p. 114). Our study contributes to elicit students’ (in our case lower secondary school students) difficulties in predicting and interpreting the hierarchical dependencies in a DGE. This knowledge may be useful to the teacher if dependency tasks are to be introduced in the classroom, or if other tasks are introduced, in which the hierarchical nature of the environment is to be exploited. Focusing on the signs emerging in the data allows not only to confirm the difficulty that students face in recognizing dependency but also show the weakness, if not the inappropriateness of the teacher. The fact that the teacher only refers to the geometrical property may limit the interpretation of the phenomenon to geometrical reasons. What can be observed is explained both by geometrical reasons and by ‘software design’ reasons. What the students are to become aware of refers precisely to both. On the contrary, we can also foresee that a fruitful intervention could start from questioning the use of specific geometrical terms, for instance, asking for a geometrical interpretation of the expression moving in parallel. The contribution of TSM in this regard, is the hypothesis about the generative role that the emerging signs may have when taken as the base for a process aiming at developing their meanings towards geometrical meanings and the awareness of the link between DGE functioning and geometrical theory.

From the case of Dan and Jan, and analysis of the data coming from other groups, we see that, in order to explain the on screen phenomena, some students refer neither to the construction process nor to geometrical properties. Instead, they use a global description of the construction, e.g. “they move along in parallel” or “the sides move along like a stick”. They seem not aware of the necessary attention to relations between local elements of the construction, and that they should interpret what happens on the screen in relation to the construction process. It may be useful to revise the task formulation concerning choices (ii-iii) to ask more directly about each element of the construction both in the pre investigation, and in the post investigation, in order to support a geometrical interpretation of the phenomenon that can be observed, i.e. instead of asking "what happens when you drag point A?", the question could be more focused, aimed at directing the attention on specific elements of the figure, e.g. "what happens **to point B** when you drag point A?" etc.

Results from task 1d show that even though the students are surprised and intellectual curiosity arises (Laborde, 2003), they may just write what happened, and quickly move on to the next task (this happened in the case of Dan and Jan). Sif and Ole are also curious and seemingly want to find out why they cannot drag point C. However, a reflection comes from the teacher's unfortunate action that seems to end the students' investigation. The task may be reformulated so that, in case the figure obtained does not behave as predicted, then the students are encouraged to explain why. This may lead us to the hypothesis that a good task choice would encourage the students observe a situation where they expect something to unfold and on the contrary, this does not happen. Afterwards they **must explain why**. The general hypothesis could be: In front of something unexpected, an explanation rises…

7. Conclusion

The emerging signs from the unfolding of the semiotic potential of DGE presented in this study, in relation to a dependency task, suggests that students do not intuitively expect the dependencies in DGE to be hierarchical. This is a useful insight if one aims to exploit the hierarchical nature of the environment for some educational objective.

The data indicates that the task formulation should be specific concerning the properties, which the students are required to focus their attention on. It seems particularly pertinent in order to direct low achieving students' attention to the dependencies between the geometrical objects that constitute the learning aim.

The task design heuristic of Predict-Observe-Explain shows promising potential as a catalyst for intellectual curiosity that can trigger the question why, however, explicitly asking the students to explain any unexpected observation seems to be necessary for active reflection.

The role of the teacher in the following phase of the collective classroom discussion was not discussed in this article, however, the episode with Sif and Ole exemplifies well how crucial the role of the teacher is. We plan to report on this important matter in future publications.

8. Declaration of interest statement

The authors declare no conflict of interest.

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Paper IV

A "TOOLBOX PUZZLE" APPROACH TO BRIDGE THE GAP BETWEEN CONJECTURES AND PROOF IN DYNAMIC GEOMETRY[1]

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The paper presents the findings of the analysis of two Danish grade 8 students working together to prove conjectures, which they formulated based on guided explorations in a dynamic geometry environment, in the frame of a design based research project. The case indicates that the designed task can bridge a connection between conjecturing activities in dynamic geometry environments and deductive reasoning. The students manage to explain theoretically, what is initially empirically evident for them in their exploration in the dynamic geometry environment. The proving activity seems to make sense for the students, as a way of explaining "why" the conjecture is true. Certain findings coming from other groups are also presented.

Keywords: Dynamic Geometry Environments, Conjectures, Proof, Toolbox puzzle approach.

INTRODUCTION AND THEORETICAL BACKGROUND

An ongoing issue in the mathematics education research field concerns the role of dynamic geometry environments (DGE hereinafter) in relation to proof. Several studies highlight the potentials of DGE in relation to development of mathematical reasoning, abilities in generalization and in conjecturing (e.g. Arzarello, Olivero, Paola & Robutti, 2002; Laborde, 2001; Leung, 2015; Baccaglini-Frank & Mariotti, 2010; Edwards et al., 2014). However, it is not clear whether such activities in DGE can support students' development of abilities in deductive argumentation. Some studies indicate that the empirical nature of the DGE investigations may impede the progression of deductive reasoning (e.g. Marrades & Gutiérrez, 2000; Connor, Moss, & Grover, 2007). That is to say, once the students have explored a construction in the DGE and discovered some relationship, they may become so convinced by the empirical experience that it does not make sense for them to prove (again) what they "know". However, other researchers suggest that students' explorative work in DGE does not have to risk development of deductive reasoning (Lachmy & Koichu, 2014; Sinclair & Robutti, 2013). Seemingly, the didactic design surrounding the work in the DGE and the role of the teacher is of utmost importance (e.g. Mariotti, 2012). De Villiers (2007) argues against a common method, which is for the teacher to devalue

[1] A short earlier version of this paper was accepted for presentation at the 14th International Congress on Mathematical Education (ICME-14).

the result of the students' empirical investigation as a means of motivating students to undertake theoretical validation. Instead, he suggests highlighting the role of proof as an explanation. The teacher may turn the theoretical validation into a meaningful activity for the students as a challenge to explain "why" their DGE investigations are true (de Villiers, 2007). Trocki (2014) suggests that motivating the students to theoretically justify their empirical explorations may also be incorporated into the task design itself.

In light of the ongoing discussion in the field on the role of DGE in conjecturing and proof, the following research question arises: *How can students' conjecturing activities in DGE be combined with theoretical validation, to make theoretical validation a meaningful activity for the students?*

This research question is investigated as a part of a larger design-based research project, in which the overarching mathematical aim is to utilize potentials of DGE in order to support students' development of mathematical reasoning competency, which is a notion from the Danish KOM framework (Niss & Højgaard, 2019). The KOM framework is a competency-based approach to describe what mathematical mastery entails, and it is integrated into lower secondary school curriculum as well as most other educational levels in Denmark. The mathematical reasoning competency includes abilities concerning reasoning, conjecturing and proving (Niss & Højgaard, 2019, p. 16). The specific designed task that is reported upon in this paper aims at bridging a connection between students' conjecturing activities in the popular DGE software, GeoGebra, and proving. However, diverging understandings exist regarding the meaning of the notion of proof in a teaching and learning context (Mariotti, 2012; Balacheff, 2008). Therefore, I will briefly impart what is implied by the notion of proof in the context of school mathematics, both in the research field and in this project.

Mariotti (2012) elaborates on different understandings of proof in school context and unfolds two extremes; 1) proof as the product of theoretical validation of already stated theorems, and 2) proof as the product of a proving process, which includes exploration and conjecturing as well as proving conjectures. Sinclair and Robutti (2013) state that the view on proof in the context of school mathematics has largely shifted to comprise proof as a process, and that this may in part be attributed to the facilitation of experimentation provided by digital technologies. The KOM framework does not address proof using the same terminology, however, proof as a process resonates with the emphasis stated in the KOM framework concerning the ability to investigate and do mathematics (Niss & Højgaard, 2019). Therefore, in this project, proof is understood as a process that includes exploration, conjecturing and deductive reasoning.

In the following sections, the method and educational context of the study is explained, followed by a description of the task design principles. Then a case is presented of two students working together on the task, followed by an analysis of the data. Finally, some conclusions are made concerning the specific case, but also referring to results coming from other groups and to research aims going forward.

METHOD

The research project is anchored in the frame of design-based research methodology (Bakker & van Eerde, 2015). Based on analysis of DGE literature, a hypothetical learning trajectory was proposed (see more in Højsted, 2019; 2020a), leading to the development of a didactic sequence that included 15 tasks. The sequence design was also influenced by results from a survey (Højsted, 2020b). The didactic sequence was tested and redesigned in three design cycles in three different schools that each lasted approximately three weeks (14-16 lessons). The data presented in this paper is from the second design cycle. To investigate the research question in this paper, a "toolbox puzzle" task was designed with the aim of supporting the students to first formulate conjectures based on guided investigations in GeoGebra, and then to undertake theoretical validation of the conjectures.

Data from each design cycle was acquired in the form of screencast recordings of the students' work in GeoGebra; external video of certain groups (chosen in collaboration with the teacher to comprise a spectrum of high-low achieving students); and written reports that were collected from the students.

In this paper, data is analysed from one pair of students, Ida and Sif, in order to investigate to what extent the toolbox puzzle design supports them in proving their conjectures and if the activity seems meaningful to them. Some results coming from other groups is also mentioned in the conclusion

EDUCATIONAL CONTEXT

The study took place in an 8th grade (age 13-14) mathematics classroom in Denmark during a period of three weeks. The students had some previous experience using the geometry part of GeoGebra, which is common in Denmark, since ability in relation to dynamic geometry programs are highlighted in the curriculum *mathematics common aims* already from grade 3 (BUVM, 2019). However, the students had no experience related to theoretical validation of conjectures or theorems, which is not surprising since it is almost non-existent in lower secondary school in Denmark, which is evident at curriculum level, in textbooks and in practice. In that light, it is no shock that Jessen, Holm and Winsløw (2015) found that Danish upper lower secondary school students lack in reasoning abilities.

TASK DESIGN

The initial tasks in the didactic sequence were designed to highlight the theoretical properties of figures, and how they are mediated by DGE in the form of invariants, e.g. by constructing robust figures in "construction tasks" (Mariotti, 2012). In subsequent tasks, the students were engaged in constructing and investigating the constructions in order to make conjectures. Generally, the design heuristic of Predict-Observe-Explain (White & Gunstone, 2014, p. 44-65) was applied to some extent in most tasks. The students were required to make a prediction concerning some geometrical properties, and to justify their prediction. Afterwards, they were to report what they observed and explain in case there were differences between prediction and observation, leading to conjectures about the geometrical constructions. Afterwards, the students were expected to explain why their conjectures were true. They were provided with a toolbox (on the right in figure 1) that contained theorems to be used in their argumentation. In the design, theoretical validation was portrayed to the students as an activity of finding out and explaining why conjectures are true, as suggested by de Villiers (2007). The toolbox was introduced to the students as a helping hand of already established truths comprising the necessary clues to find out why their conjectures were true, much like pieces to solve a puzzle.

The task[2] reported upon in this paper consisted of an initial construction part, followed by questions (Predict-Observe-Explain) to guide the students to discover and make a conjecture about the relationship of an exterior angle of a triangle with its interior angles. Finally, the students were encouraged to explain/prove the conjecture in a proof sheet (on the left in Figure 1), using a toolbox, which contains a support drawing as well as information (angle over a line is 180°, and the angle sum of a triangle is 180°) to be used in the argumentation.

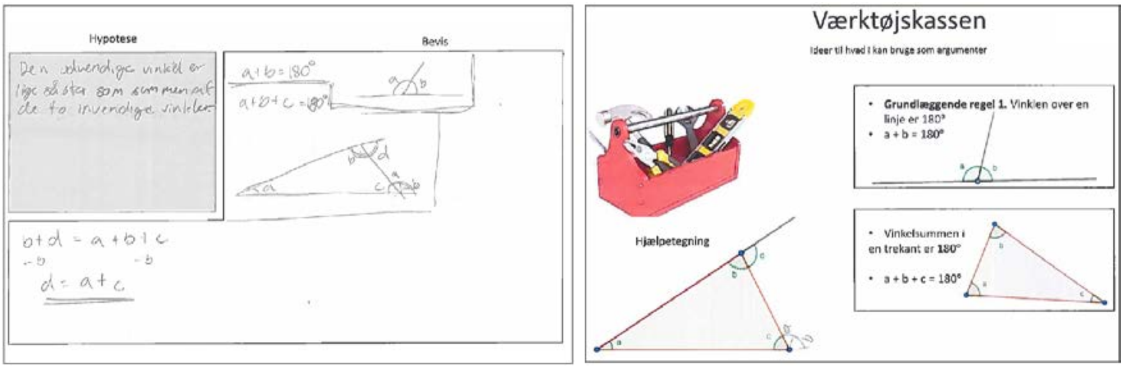

Figure 1: The proof sheet and toolbox. Solved by Ida and Sif

THE CASE OF IDA AND SIF

Ida and Sif were described by their teacher as medium to high achieving students. In the previous task, they found the proving activity and the toolbox to be confusing.

[2] Task 9**a**. Construct an arbitrary triangle and extend one of the sides. **b**. What is the relationship between the exterior angle *c* and the interior angles *a* and *b*? Guess first before you measure! [*There is a figure with the mentioned angles on the task sheet*]. **c**. Measure the angles and find the relationship. **d**. Drag to investigate which situations the relationship applies to. **e**. Discuss with your partner and make a conjecture about the relationship between the exterior angle and the interior angles. **f**. write the conjecture in the proof sheet. **g**. You can see in GeoGebra that it is true, but can you explain why it is true? Use the information from the TOOLBOX to argue.

The following excerpt ensues after Ida and Sif have constructed the figure from task 9, they have guessed, investigated and put forward the correct conjecture (9a-9f) and are about to try to explain/prove why it is true (9g):

516	Ida	The sum of the two interior angles… [*Writes the conjecture in the proof sheet (Figure 1) translated: "The external angle is as large as the sum of the two internal angles"*]
517	Sif	Beautiful! Okay, now we have to prove it. Oh no…
519	Sif	Now that again…
520	Ida	*a* plus *c* equals *b*, and see. Basic Rule 1: The angle over a line is. The angle sum of a triangle is. [*reading from the tool box*]
521	Sif	Yes! I understand. Look… [*points to the support drawing in the tool box*]
522	Ida	Ohh.
523	Sif	Super! In here, that's what's missing. [*points to angle b in the support drawing (see figure 2)*]

Figure 2: using the toolbox to explain

…

533	Sif	And add this one here, to here. [*pointing to angle b being added to a+c and to d respectively*]
534	Ida	That's right, so it makes 180. AND it makes sense. Is there more to say?
535	Sif	That is… just how it is.
536	Ida	We know that the sum in a triangle is 180 degrees and that the sum… the angle sum of a line is 180 degrees. Therefore, when we are missing an angle here…

In the events that follow, they write their answer (Figure 1), but express difficulty in doing so, because they expect that they must use algebra in their answer:

574	Ida	How do we write that in mathematical language?

…

595	Ida	Ah okay! And *a* plus *b* and *c* yes. And *b* plus *d* it also gives 180
597	Sif	This one plus this one, is the same as these three. [*pointing to b+d and a+b+c*]
598	Ida	That's right. It's actually right. Oh, *b* plus *d* equals *a* plus *b* plus *c* because this makes 180, and this makes 180.

Analysis

We can notice from lines 517-519 that Sif is not excited about the prospect of having to prove the hypothesis. In fact, it was observed in several groups, that the activity of theoretical validation was not enthusiastically undertaken immediately. It was also evident, that the proving part was the most challenging part of the task, which may partly explain the lack of enthusiasm. However, the mood towards the proving activity changed in the case of Ida and Sif, and in some other groups as well, when they had worked on 2-3 tasks of this type, which indicates that they had to get accustomed to the task design. Some of the difficulty may be attributed to the openness and unfamiliarity of the answer format, since several students could put forward their reasoning verbally, but struggled to write down their argumentation. Ida and Sif also struggle with this issue (line 574-590). However, they find it easier to write the answer in subsequent tasks, after the teacher explained that they could write their arguments using natural language narratives.

In line 520, we see that Ida immediately turns to the toolbox information, reading aloud the two pieces of information provided, which indicates that she has realised the usefulness of the toolbox. Sif listens and seems to recognize that adding angle b to $a+c$ and d respectively in both cases gives 180° (line 521-533), which she manages to support Ida to grasp and elaborate as well (line 534-536). They manage to reason deductively that their conjecture is valid, and after some struggle, write their answer algebraically (Figure 1). The sequence of utterances from the students indicate that it is a sense making activity for them, and that there seems to be intellectual satisfaction attached to their experience (line 534-536).

CONCLUSION AND FORTHCOMING REPORTS

The study indicates that the “toolbox puzzle” approach can bridge a connection between conjecturing activities in DGE and deductive reasoning. The students explained theoretically, what they initially guessed purely visually and secondly investigated empirically in DGE. Importantly, the activity of conducting the theoretical validation seemed to make sense to them.

It was apparent that Ida and Sif had to become acquainted with the structure of the toolbox puzzle approach, before it became a sense making activity for them. This point was also evident in other groups. Additionally, several groups found it difficult to write down their arguments even though they could convince each other verbally and with the help of gestures.

Most groups of students succeeded and seemed to enjoy the exploration and conjecturing part of the tasks in the sequence. However, medium-low achieving students struggled to string together coherent deductive reasoning, and some never managed to overcome the toolbox puzzle part of the task on their own.

Other aspects of interest in this study is to what degree the students use DGE as they are trying to make a deductive argument, and what role the DGE plays in this regard. There are some indications that the students go back to the DGE in order to exemplify arguments to each other. Notably, early analyses also show that some students return to DGE in order to verify what they have proven(!). In that case, even though proof as an explanation makes sense to the students, it does not highlight the status of their product. I.e. the value of theoretical validation is not yet appreciated. This interplay between the theoretical validation and ensuing DGE actions will be the focus of attention in the ongoing project, which I hope to report on in future publications.

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Paper V

Guidelines for the teacher – are they possible?

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We report on the design and implementation of teacher guidelines in a didactic sequence that was developed in the frame of a design-based research study. The guidelines are anchored in the frame of the Theory of Semiotic Mediation and the hypothesis was that the guidelines could support the teachers' management in the classroom discussions. We elaborate on the teacher guidelines design and present results of analysis from a classroom discussion as well as an interview with the teacher. Our results show an implementation that was not particularly effective, and that the teacher followed the guidelines only to some extent. We discuss the dilemma of developing condensed guidelines while trying to convey underpinnings of an elaborate theoretical frame.

Keywords: Teacher guidelines, dynamic geometry environments, the theory of semiotic mediation.

Introduction and conceptual frameworks

Since the introduction of dynamic geometry environments (DGE), ample research has been conducted to outline the potentialities of the software to foster students learning of mathematics (for an overview see Højsted, 2020). While there are many studies that have shed light on the students' work in DGE from a cognitive point of view, not least in relation to mathematical reasoning, generalizations and conjecturing (e.g. Arzarello et al, 2002), there has been less research conducted in relation to the design of adequate tasks to utilize the potentials of DGE for specific mathematical learning aims, as well as the on the role of the teacher in the mathematics classroom in which the DGE potentials are to be utilized (Højsted, 2020; Komatsu & Jones, 2018). A promising framework that acknowledges the essential role of the teacher in artefact-based activities is the Theory of Semiotic Mediation (TSM) (Bartolini-Bussi & Mariotti, 2008). Taking a semiotic perspective, the TSM provides a model of the teaching and learning process centered on the role that a specific artefact can play because of its semiotic potential with respect to a specific mathematical meaning. The semiotic mediation process develops form the use of the artefact and students' personal meanings emerging from that, towards the evolution of shared mathematical meanings, which is accomplished in a collective discussion managed by the teacher, who is expected to play a crucial role. The TSM frame describes organization of the teaching/learning sequence according to iterate didactic cycles, which comprise (1) activity with the artefact (2) production of signs (writing etc.), followed by (3) classroom discussion.

The paper reports on part of a project whose overarching aim is to develop guidelines for the design of didactic sequences that utilize the potentials of a specific artefact, a DGE in relation to fostering students' development of mathematical reasoning competency (Højsted, 2020; Højsted & Mariotti, 2020). When we use the notion of mathematical reasoning competency, we refer to the Danish KOM framework description of reasoning competency (Niss & Højgaard, 2019).

In this paper, we report on our attempts to convey the rationale of the TSM by developing **guidelines for the teacher**. In particular, considering the delicate proactive role that the teacher is asked to play when managing a collective discussion, the guidelines intended to outline a general frame for the teacher's purposeful interventions. Such a frame, empirically developed (Mariotti, 2013), consists of four categories of actions that the teacher can perform with the aim of fostering the process of semiotic mediation centered on the use of a particular artifact.

The **back to the task** action asks the students to report on what they did. "The objective is that of reconstructing the context of the artefact" in order to make meanings "emerge in relation to that experience." (Bartolini-Bussi & Mariotti, 2008, p. 775). 2. **Focalize** is complementary to the previous action, and aims at focusing on relevant aspects of task experience with respect to the intended mathematical meaning. 3. **Ask for a synthesis** action aims at fostering the move towards mathematical meanings by asking students to generalize and decontextualize the experience from the specific artefact task. 4. **Give a synthesis** complements the previous action and is used to support students' synthetizing.

The interventions of the first pair of categories refer to the unfolding of the expected semiotic potential of the given artifact, and aims at fostering students move from their experience with the artifact towards pertinent aspects that can be related to the mathematical meanings that constitute the educational goal. The second pair of interventions refer to the move towards the introduction of the expected mathematical meanings.

In a previous experience (Højsted & Mariotti, 2020), students showed difficulties in grasping the hierarchical dependencies between objects in GeoGebra, while the teacher seemed not aware of this fact, and therefore could not help the students to interpret the onscreen phenomena sensibly.

Considering that the teacher could not manage to help the students overcome this difficulty, we designed guidelines for the teacher, by implementing the didactic cycle and the teacher actions from the TSM frame. Our goal was that of providing the teacher support to interpret students' difficulties and to intervene to make them overcome them. We propose the guidelines as the product of a design process, making explicit the principles of design and reporting on the a posteriori analysis of the implementation in the classroom. The analysis of the collected data is aimed at studying the effect of these guidelines. In this paper, we are therefore focused on answering the following research question:

In what ways do the teaching guidelines support the teacher in holding classroom discussion; to what extent is the support consistent with our expectation; and based on the collected data, which revisions can we propose?

We set out by briefly reporting on the methodological approach; the designed task and teacher guidelines; as well as the data collection. In the following results and analysis section, we look at the first classroom discussion after the students have worked on tasks 1 and 2 and at data from the subsequent interview with the teacher, in order to identify to what extent the teacher guidelines were used, and find out to what extent they were useful for the teacher. On this basis we suggest revisions and consider emerging issues in the concluding discussion.

Method, design and data collection

This study is part of a design-based research project, which is methodologically characterized by its dual purpose of developing both educational practice and theory about practice, by means of iterations of design and testing of educational activities (Bakker & van Eerde, 2015).

Based on an a priori analysis of the semiotic potential of DGE in relation to reasoning competency (Højsted, 2020), a didactic sequence was developed and tested in three iterations in Danish 8th grade (age 13-14) classes (Højsted & Mariotti, 2020). For the last iteration, we designed guidelines for the teacher. The criteria of design was to include the explication of key elements of TSM rationale: specifically, the notion of didactic cycle, of semiotic potential of a DGE in relation to reasoning competency, and specific instructions about the management of the collective discussion, e.g. the four actions described above. We opted for a concise text written in a simple language, refraining from using too many technical terms.

We met with the teacher prior to the experiment to discuss the guidelines, attempting to share hypotheses and design principles of the teaching material, which included tasks for the students, a task answer book for the teacher, and the guidelines for the teacher.

The task and the teacher guidelines

In this paper, we discuss data coming from the classroom discussion following the very first task, the “dependency task” (Højsted & Mariotti, 2020): 1.a. Construct two points A and B in GeoGebra and the midpoint C between them. Use the midpoint command […] 1.d. What do you think happens with the other points when you drag point C? Guess and justify first. Investigate afterwards, what happens?

As mentioned, *the teacher guidelines* consisted of a two-page introduction describing the general structure of a didactic cycle and in particular, elaborated on the four categories of actions the teacher was expected to perform. The specific instructions that followed were organized into three main constituents, presented in a table: 1. a description of the educational aim of each task, 2. the personal meanings expected to emerge from the activity (hypothetical and experienced in previous design iterations), 3. and the corresponding possible teacher actions that the teacher is advised to perform, followed with specific examples and comments.

The first classroom discussion was planned to take place after the students had worked on task 1 and 2, marking the end of the first cycle. The guidelines advised the teacher to perform the first pair of actions (back to the task and focalize) in relation to each subtask, and then after discussing the whole of tasks 1 and 2, to ask for a synthesis and give a synthesis. We provide a sample from the table.

Task no.	Purpose of the task (the educational aim)	Expected student meanings	Possible teacher actions in the discussion
…	…	…	…
1d	Specific sub-goal: to become aware that in GeoGebra the derived points cannot affect the free objects that have defined them. I.e. that C cannot be moved because if you drag C, then A and B have to follow in order to maintain C as the midpoint, but this is not allowed	Some students guess/expect that point C can be moved by direct dragging and that the other points will follow it as if the figure had a rigid/solid structure. Possible student guesses "They move together", ”*A and B follow in parallel* ".	**Back to the task** They probably won't say why C can't move. **Focalize** Therefore, ask the students about the construction process. How did point C come into the world? In this way, the focus can be oriented

	in GeoGebra. A derived object (a child) cannot affect the objects from which it is defined (its parents). C is therefore a locked point. In contrast, derived objects can be moved if they do not affect the objects that have defined them, as we see in task 2.	They will probably be surprised that one could not move C. And they will probably have a hard time explaining why, but possibly someone mentions that C cannot be moved because it is made based on the others. This can be highlighted in the discussion.	towards C being a derived object from A and B. The terminology of children and parents can be used. C is a child of A and B (its parents). In GeoGebra you cannot move objects (children) that are derived from other objects (parents), if that requires that the parents follow.
…	…	…	…
After 1-2	The specific objectives in the sub-tasks above contribute to the overall purpose of tasks 1 and 2: Understanding that it is dependency relations between geometrical properties of objects in GeoGebra that determine the outcome of dragging, and that these dependencies stem from the construction method, and its logical consequences from the geometric rules that govern the program. And that these relationships remain when you drag points. The goal is for students to describe the objects and dependencies in geometric terms. That they can justify what they see based on the objects, their dependency relationships and the construction method.	After task 1 and 2, the students are expected to be thinking about the relationships in the constructions and to connect those with construction method. I.e. that they can describe how the figure behaves in GeoGebra referring to geometric properties.	**Ask for a synthesis** For example, ask students what determines how a figure moves in GeoGebra, try to get them to say something general about the construction process and geometric properties. **Give a synthesis** Offer that there are dependency relationships between the objects in GeoGebra that determine the specific points' behavior, and that these relationships stem from the construction method that induces logical consequences based on the geometric properties defined by construction.

Table 1: The table from the guidelines for the teacher to hold classroom discussions.

Our hypothesis was that the guidelines would support the teacher to hold effective classroom discussions and we expected the teacher to use the advice put forward in the table.

Data collection and analysis

The classroom data was collected in the form of video of the whole class and selected groups; in screencast recordings of all groups; audio recordings; and written products from the students. Data was also collected by interviewing the teacher before and after the teaching experiment and after each teaching session. Through a semi-structured interview, we asked the teacher about the guidelines, and the perceived usefulness of the guidelines from the point of view of the teacher.

Results and ensuing analysis

The teacher opened the discussion asking students to report on their responses to subtasks 1a-1c, which seemed straightforward for the students. GeoGebra is running on the whiteboard with the construction from the task performed. The discussion goes on until they reach subtask 1d. The teacher, apparently expecting the task to be more difficult, seems excited to discuss this task.

172 Teacher: So I'm a little excited about this with point C, what do you think happens to points A and B when you drag in point C. What did you guess would happen here? Freja.

173	Freja:	We guessed that nothing happens.
174	Teacher:	That nothing happened at all?
175	Freja:	No, we just thought that if it was that you dragged C that, either it would move, but we did not think it would get bigger, or that the C would not move.
176	Teacher:	Well so, if I drag this point, it would all move as if it were one line? *[She simulates grabbing point C with her hand at the whiteboard, and simulates dragging it up and down]*
177	Freja:	Yes, or it would remain as it is.

The teacher starts with prompting students' guesses (172), but though apparently using a **back to the task** action, the intended aim of this action – fostering students to express personal meanings – is not fulfilled. It is herself that explains the task formulation, and the following interactions show her gently but firmly guiding the discourse; she reiterates what the students say, (174, 186, 198, 219, 223, 230), but changing her tone to a question tone provides an implicit evaluation of the students' suggestion. Freja seems to put forward two scenarios, it will not move, or it will move as a solid. The teacher simulates what Freja says, using the construction on the whiteboard (176). The teacher wants more signs to emerge, and asks others.

185	Maja:	Yes, so if you moved it up [point C], it would just straighten up like a triangle but still in the middle.
186	Teacher:	And get a triangle out of it, yeah okay interesting.

…

197	Freja:	And the C it will get out of control.
198	Teacher:	It will get completely out of control. Yes. Okay! Interesting, so there were actually more suggestions here, what happened when you dragged it. Dima?

Maja's guess indicates some misunderstanding of midpoints, while Freja's guess indicates that they did not have any explanation. To both answers, the teacher merely replies that it is interesting, and moves on, now to ask another student what actually happened. Several students describe that point C cannot be dragged. The teacher stops on this and asks students for a justification.

217	Teacher:	Why do you think that is? Why can't I do something about it Julie.
218	Julie:	Yeah because that's no point.
219	Teacher:	That is because it's no point. Okay, can you try to say a little more about that it's not a point. What is the difference between points A, B and then C?
220	Julie:	Because that's the midpoint.
221	Teacher:	Yes, but there's one such dot right there, so there's a point?
222	Julie:	Yes, but it's not blue.
223	Teacher:	It's not blue, okay.

The student has noticed that point C is different than A and B, it even has a different color. When Julia says it is not a point, the teacher asks her to elaborate, helping her by asking what is the difference between the three points (line 219). This can be considered the start of a **focalization** action. However, she does not ask about the construction process, trying to foster awareness of derived objects, which was the advice given in the guidelines. Instead, she just asks why they cannot drag point C. The teacher asks others for more suggestions.

229 Tobias: That is, A and B are points that you have made.
230 Teacher: It's something I have made.
231 Tobias: Yeah, you didn't make point C.
232 Teacher: Yes, okay. But I actually have, because I used that tool after all. Larso!?
233 Larso: I think point C it just simulates the center of both points.
234 Teacher: Yeah, okay. That's the same thing you would say Maja?
235 Maja: Yes.

Tobias suggests A and B were generated by the user, in contrast with the fact that the user did not generate C directly. The utterance "have made/didn't make" could develop into the categories "Free points" / "derived points" and then into the relationship that derived points depend on free points. However, the teacher is not able to manage this evolution; she stops the semiotic process by talking about the use of the tool (232) in a manner that invalidates Tobias' suggestion. Several students suggest that C is different to A and B (218, 220, 222, 231, 233, 235). The teacher did not perform further **focalizing** actions; thus, the students' suggestions remained suggestions. Without reaching a consensus in class, or an accurate explanation, the teacher just moves on to the next task. It seems that the teacher has no clear view of the objective of the discussion.

After discussing tasks one and two, the teacher did **not** ask for a synthesis or give a synthesis herself, even if the teacher guidelines advised her to do so. It seems she does not feel the need of that.

Teacher interview

Actually, in the interview after the teaching session, when asked about the teacher actions, the teacher seems to realize the synthesis was missing.

114 Researcher: Were there any of the four teacher actions that you used, or perhaps used the most? And were there some that you didn't use so much?
115 Teacher: Ehm, I don't know if…
116 Researcher: You mentioned that one "back to the task" before.
117 Teacher: Yeah, that one I think I managed to use it quite well. To ask "what was the purpose", "what have you investigated", "ok, what was actually the purpose", "what is our conclusion then", or like that. Ehm… Yeah I don't know… I think maybe I, except that one about give a synthesis, I think I used the others pretty much.

During the classroom discussion, the teacher held the task answer book, and not the guidelines. She had read the guidelines beforehand, and decided that it was more important to have the answer book.

38 Teacher: But that is mostly because when I am standing there, then I have to juggle, suddenly they have said something, and then I am about to confirm something wrong, then it is very nice to stand with the answer book. And if I should stand with both, then I would drown in papers, right.

When asked about the usefulness of the guidelines, the teacher described that she had found them useful. The teacher apparently felt she had followed them, especially the general description of the four teacher actions.

62 Teacher: It was perhaps some of the general, so the general setup [pointing to the first part of the teacher guidelines that contain the four teacher actions] […]

She indicated that the guidelines table contained too much information, and suggested that there should only be a few examples of questions to pose, in addition to the general teacher actions.

70 Teacher: For sure in the following guidelines, I think, the first two [tasks] can be very detailed, but in the following, just a few proposals for questions […]

Since the teacher did not use the questions or the terminology from the guidelines table during the classroom discussion, the question arises whether or not she had fully read them. Alternatively, perhaps she had forgotten some parts or did not understand everything, and since she did not look at the guidelines during the classroom discussion, she was unable to use them effectively.

Concluding discussion

From the data, we can conclude that only parts of the guidelines were acknowledged and only parts of the suggested interventions were performed. This is the case for the first pair of actions categories, **back to the task** and to some extent **focalize**, however, not for the second pair of actions, which were completely neglected. When guideline suggestions were followed what seems missing are some aspects of the general aim of the actions. For instance in the case of back to the task, although the teacher encourages students to express themselves, it seems difficult for her to let students elaborate on their own formulation without intervening and she does not leave the students to talk to each other. There is always an intervention of the teacher in response to the intervention of a student in a "ping-pong" effect. The teacher recognizes that reacting 'on the spot' is difficult for her because she does not feel certain about the answer (line 38), which unveils that she feels that she has to confirm, or reject, what is said. Actually, this can explain the 'ping pong' pattern of the discourse; the reaction corresponds to the feeling of obligation for confirming. We wonder if the teacher would behave differently if explicitly advised to keep silent and wait for the students to fuel the conversation. Another example of missing aspects of the general aim is in the case of Tobias (229-232).

The teacher indicates that the guidelines need to contain less information to be effective (line 70). A possible revision of the guidelines could consider condensing and incorporating the guidelines table into the answer book, giving more examples of questions and statements related to the four teacher actions for each task (or set of tasks). However, it is questionable if further condensing the guidelines would be helpful: in fact, the actions of asking for a synthesis and giving a synthesis seem not being appropriated by the teacher. From the point of view of the TSM frame, reaching a synthesis shared and elaborated by the whole class is a crucial part of the teaching and learning process.

This leads us to a conclusion that emerges as a critical issue - to what extent should the theoretical frame be made explicit in the guidelines in order to give the teacher the conceptual tools to interpret the guidelines? If the teacher does not share the rationale of the TSM framework, it may happen that she disregards some key aspects, because she does not grasp their importance. We did not incorporate much of the underlying theoretical assumptions of the TSM frame in the guidelines, so we can make the hypothesis that it is one of the reasons the teacher did not use much of the guidelines. We suspect that the teacher was too far from this pedagogical perspective to really appropriate it.

The dilemma is that the teacher already suggests that the guidelines must be shorter, while the data suggests that more of the theoretical frame must be shared with the teacher, and the importance of key aspects must be elaborated.

Seeing as the teacher only followed the guidelines to some extent, neglecting two of the teacher actions, and did not use the advice from the table, we can conclude that it is difficult to communicate theoretical aspects in the form of guidelines, at least in the chosen design form of a condensed text and the table. Perhaps the guidelines somehow need more flexibility to be adapted to different teachers' pedagogical paradigms. Conversely, we can notice the important need of teachers' flexibility and pedagogical awareness in presenting new activities with technological tools.

More insights are needed on designing effective guidelines, which may be considered the core of the articulation between theory and practice. It requires reflecting on how to interface with teachers taking into account the diversity of their possible pedagogical paradigms, most of the time implicit. We intend to report more on this matter in forthcoming research.

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